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Quantum Space–Time and Gravitational Consequences

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Relativistic particle dynamics and basic physical quantities for the general theory of gravity are reconstructed from a quantum space-time point of view. An additional force caused by quantum space-time appears in the equation of particle motion, giving rise to a reformulation of the equivalence principle up to values of $O(L^2)$, where L is the fundamental length. It turns out that quantum space-time leads to quantization of gravity, i.e., the metric tensor $g_{\mu\nu}(\hat{z})$ becomes operator-valued and is not commutative at different points x^μ and y^μ in usual space-time on a large scale, and its commutator depending on the "vielbein" field (gaugelike graviton field) is proportional to L^2 multiplied by a translation-invariant wave function propagated between points x^μ and y^μ. In the given scheme, there appears to be an antigravitational effect in the motion of a particle in the gravitational force. This effect depends on the value of particle mass; when a particle is heavy its free-fall time is long compared to that for a light-weight particle. The problem of the change of time scale and the anisotropy of inertia are discussed. From experimental data from testing of the latter effect it follows that $L \lesssim 10^{-22}$ cm.

1. INTRODUCTION

One of the important unsolved problems of quantum field theory is the quantization of gravity. Recently, this has become more pressing in light of the development of unified ways of describing the fundamental forces in Nature. Although the force of gravity is extremely weak with respect to electromagnetic and weak (or electroweak) and strong forces between elementary particles, it is still nonzero, so that as increasing energies probe deeper and deeper into matter, a level eventually should be reached where quantum gravitational effects show up. This is the so-called "Planck mass" of about 10^{19} GeV.

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Many attempts have been made to construct the quantization of gravity from different points of view (see, for example, Markov et al., 1985; Sato and Inami, 1986), among which the ideology of supersymmetry and superstring theory is the most attractive and gives rise to the hope that a quantum picture of gravity can be described by analogy with other field interactions. Contrary to the usual approach, within the framework of the latter, forces are not interpreted as the interaction of pointlike particles, but are regarded as infinitely small, creeping and zigzag-like one-dimensional strings. According to E. Witten, this method may lead to a new understanding of the space-time structure of the general theory of gravity.

In a series of papers we want to consider the problem of the quantization of gravity within quantum space-time with coordinates z^{μ}:

$$z^{\mu} = x^{\mu} + L\Pi^{\mu}(x) \tag{1}$$

where the $\Pi^{\mu}(x)$ are arbitrary noncommutative matrix functions of the points x^{μ}. One possible interpretation of quantum space-time with (1) can presumably be found by taking x^{μ} as usual coordinates and interpreting $\Pi^{\mu}(x)$ as additional (non-Abelian-like) fields. Such a type of space (1) with some applications has been discussed in earlier work (Namsrai, 1985; Dineykhan and Namsrai, 1985a). The present paper has an introductory character, giving mathematical methods, defining basic quantities of the theory, and formulating its principles. We also consider some physical consequences of the results.

The choice of coordinate form (1) for quantum space-time is based on the following argument: from high-energy physics and other physical considerations, especially from astrophysical data (see below), it is well known that the parameter L of the theory (which characterizes the domain within which the quantum space-time structure or quantum gravitational effect may be manifested) is a very small value of the order of $L \lesssim 10^{-16}$-10^{-17} cm or even $L \lesssim 10^{-20}$ cm (see also Bracci et al., 1983). Therefore, in practical applications of the theory, the quantum picture of space-time and gravity may be considered as a small background perturbation effect (background radiation field) over the entire continuous space-time, or in other words, contributions to observable effects or any physical processes due to the expected quantum structure of space-time are indeed very small. This tells us that if the quantum nature of space-time exists in Nature, its structure should be negligibly small at presently attainable energies. But when energy is increased, expected structural effects become more and more sensible. Thus, we suggest that space-time with quantized coordinates is slightly different from classical continuous space-time, i.e., its coordinates may be formed by means of formula (1). In this view, the continuum theory is considered to be an approximation up to (at least) the order of $O(L^2)$.

Therefore, our aim is to calculate corrections to the continuum theory when the parameter L is small but nonzero.

In our model, yet one more question arises: how to construct a continuum theory with fundamental length by using a quantum one. It is a common problem for all schemes devoted to the construction of quantum gravitational theory. This problem is connected directly with the procedure of passage from quantum space-time in the microworld to nonquantized space-time on a large scale. This procedure requires some mathematical methods depending on the concrete realization of the introduction of quantum space-time into physics. We realize this program by two steps: first, we introduce quantum space-time with coordinates (1) as a transformation of one (usual) coordinate system to another (quantum) and vice versa, and construct physical quantities under this "quantum" transformation law. As a result, the physical theory of space-time depending on noncommuting variables becomes a quantum one involving the parameter L inexplicitly. Second, according to the assumption that the value of L is very small and that all observable physical processes may be understood as averaged values over the background of quantum space-time, one can obtain physical quantities in it by means of expansion over the parameter L, keeping terms up to the order of $O(L^2)$, in which we carry out some averaging procedure, which reduces to taking the trace of Π^{μ} matrices or making use of the expectation value over background (or base-vacuum like) states in quantum space-time. In this way, physical quantities in quantum space-time are constructed by using the correspondence principle, according to which the usual theory is obtained in the limit $L \to 0$.

In this way we have succeeded in defining the dynamics of relativistic particles and in extending the given method to the theory of gravity in quantum space-time. Thus, we observe that if space-time has quantum nature at small distances, then an additional force caused by this structure inevitably appears in the particle dynamics. This allows us to reformulate the equivalence principle in the general theory of relativity to an accuracy with terms of the order of $O(L^2)$. This means that the basic principles of the theory are slightly violated at very small distances, and that anisotropy of inertia takes place everywhere and is proportional to L^2 terms, i.e., in quantum space-time it leads to another law of particle motion with respect to that in inertial systems of reference. This new dynamical law gives rise to the appearance of an antigravitational effect in the motion of a particle in the gravitational force. In this sense, quantum space-time may be regarded as a source of a fifth force in Nature, the occurrence of which is more sensible in the microworld. Moreover, our method turns out to be useful in the quantization of gravity. As a first step, we define here the metric tensor $g_{\mu\nu}(z)$ and its commutator. Further, by the affine connection method we find the equation of particle motion in presence of gravity and show a connection between $\Gamma^{\lambda}_{\mu\nu}$ and $g_{\mu\nu}$ in the quantum space-time case. We also consider some physical consequences of the theory in order to estimate the value of the fundamental length.

In Section 2, we introduce quantum space-time into the theory for the Minkowski space-time case and define proper time by means of which four-velocity and -force are calculated. Here the nonrelativistic limit is also considered (for details, see Dineykhan and Namsrai, 1985b). An additional force is determined by using the action principle. A nonrelativistic quantum mechanical formalism is also discussed in order to estimate the change of energy levels of hydrogenlike atoms in quantum space due to an additional force. Section 3 deals with the formulation of the equivalence principle in quantum space-time and also with the quantization problem of gravity. Here primary attention is paid to the mathematical definition of the theory within the transformation method of coordinate systems. By using a tetrad formalism, we calculate explicitly the commutator of the metric tensor at different points of usual space-time on a large scale. The motion equation of the particle in the gravitational force is investigated and the affine connection method for quantum gravity is defined. Section 4 is devoted to the study of concrete applications of the formalism. Here we discuss the change of time scale and the anisotropy of inertia due to the quantum nature of space-time and explain the antigravitational effect on particle motion in an external gravitational force.

2. RELATIVISTIC PARTICLE DYNAMICS IN QUANTUM SPACE–TIME

2.1. Proper Time Formalism in Quantum Space–Time

Basic dynamical quantities, such as velocity, acceleration, and force, should first be defined by the correspondence principle. For example, by definition the velocity of a particle in quantum space-time is given by

$$U^{\mu} = dz^{\mu}/ds_q \tag{2}$$

where the variables z^{μ} are defined by (1). The symbol ds_q in (2) is to be understood as follows. First of all, we notice that the definition of the proper time variable $d\tau = ds/c$ in the quantum space-time case should be generalized, where $ds^2 \equiv dx_0^2 - d\mathbf{x}^2 = c^2dt^2 - d\mathbf{x}^2$ is the interval between two events situated infinitesimally near each other. Thus,

$$d\tau_q = ds_q/c = (1/c)(dz^2)^{1/2} \tag{3}$$

where

$$dz^2 = \eta_{\mu\nu}\, dz^\mu\, dz^\nu = dz^\mu\, dz^\mu$$

$$\eta_{\mu\nu} = \begin{cases} -1 & \nu = \mu = 1, 2, 3 \\ 1, & \nu = \mu = 0 \\ 0, & \nu \neq \mu \end{cases}$$

is the Minkowski metric. Sometimes (for example, in this section) we explicitly retain the velocity of light c in formulas. But in any case, we use, as usual, the system of units in which $c = 1$. We now calculate proper time given by formula (3) in quantum space–time. In accordance with (1), we have

$$d\tau_q = \frac{1}{c}\left[\eta_{\mu\nu}\left(dx^\mu + L\frac{\partial \Pi^\mu}{\partial x^\rho}\, dx^\rho\right)\left(dx^\nu + L\frac{\partial \Pi^\nu}{\partial x^\delta}\, dx^\delta\right)\right]^{1/2}$$

$$= \left[\frac{dx^\mu}{c}\,\frac{dx^\mu}{c} + \frac{2L}{c^2}\,\frac{\partial \Pi^\mu}{\partial x^\rho}\, dx^\rho\, dx^\mu + \frac{L^2}{c^2}\left(\frac{\partial \Pi^\mu}{\partial x^\rho}\, dx^\rho\right)^2\right]^{1/2}$$

or its ratio with respect to the time interval dt in the system of reference where a clock is rest is given by

$$\left(\frac{d\tau_q}{dt}\right)^{-1} = \left[V^\mu V^\mu + 2L\frac{\partial \Pi^\mu}{\partial x^\rho}\, V^\rho V^\mu + L^2\left(\frac{\partial \Pi^\mu}{\partial x^\rho}\, V^\rho\right)^2\right]^{-1/2}$$

$$= \beta + \delta \tag{4}$$

where

$$\delta = -\frac{1}{2}\beta^3\left[2L\frac{\partial \Pi^\mu}{\partial x^\rho}\, V^\rho V^\mu + L^2\left(\frac{\partial \Pi^\mu}{\partial x^\rho}\, V^\rho\right)^2\right] + \frac{3}{2}\beta^5 L^2\left(\frac{\partial \Pi^\mu}{\partial x^\rho}\, V^\rho V^\mu\right)^2$$

$$V^\rho = \{1, \mathbf{v}/c\}, \qquad \beta = (V^\mu V^\mu)^{-1/2} = (1 - \mathbf{v}^2/c^2)^{-1/2}$$

$\mathbf{v}$ is the usual velocity of the particle. Thus, we see that in quantum space–time, proper time depends on the fundamental length, is complicated, and has operator-valued structure. Therefore, in order to obtain its value over the large scale of space–time some mathematical procedure of averaging should be carried out. This procedure is reduced to taking the trace when matrix functions $\Pi^\mu(x)$ are given by the tetrad formalism: $\Pi^\mu(x) = \gamma^a e_a^\mu(x)$ ($a = 0, 1, 2, 3$), where γ^a and $e_a^\mu(x)$ are Dirac matrices and "vielbein" or tetrad fields, respectively. Here the question arises of how to choose the metric form in internal tetrad space–time. There are two possibilities: either Pauli's or Feynman's metric forms can be used. The cases differ when we take the trace of γ^a matrices. For the former and the latter case, we have

$$Sp(\gamma^a\gamma^b) = 4\delta_{ab}, \qquad Sp(\gamma^a\gamma^b) = 4\eta_{ab}$$

respectively. In this paper, we consider both cases separately, giving completely different results. From the physical application point of view, both cases are very interesting. Thus, for the second case, after taking a trace, we get

$$\Delta\tau_q = \langle d\tau_q\rangle = \frac{1}{4}\, Sp(d\tau_q) \approx dt\left(\beta^{-1} + \frac{L^2}{2}\,\beta I_1 - \frac{1}{2}\,L^2\beta^3 I_2\right)$$

where

$$I_1 = \eta_{ab}\frac{\partial e^\mu_a}{\partial x^\rho}\,V^\rho\,\frac{\partial e^\mu_b}{\partial x^\delta}\,V^\delta, \qquad I_2 = \left(\frac{\partial e^\mu_a}{\partial x^\rho}\,V^\rho V^\mu\right)\left(\frac{\partial e^\nu_b}{\partial x^\rho}\,V^\rho V^\nu\right)\eta_{ab}$$

For simplicity, we consider two-dimensional space–time and a simple wave-like form for $e^\mu_a(x)$,

$$\begin{pmatrix} e^0_0 & e^0_1 \\ e^1_0 & e^1_1 \end{pmatrix} = \begin{pmatrix} \sin(\omega t - kx) & \cos(\omega t - kx) \\ -\cos(\omega t - kx) & \sin(\omega t - kx) \end{pmatrix} \tag{5}$$

Then, it is easily verified that

$$\begin{aligned} I_1 &= 2\cos 2(\omega t - kx)(\omega - kv)^2 c^{-2} \\ I_2 &= \tfrac{1}{2}\, I_1 - 2vc^{-3}(\omega - kv)^2 \sin 2(\omega t - kx) \end{aligned} \tag{6}$$

As a result, in quantum space–time the proper time of the particle oscillates

$$\begin{aligned} \Delta\tau_q = dt\Bigg\{&\left(1 - \frac{v^2}{c^2}\right)^{1/2} + \frac{L^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \cos 2(\omega t - kx)(\omega - kv)^2 \\ &- \frac{L^2}{2c^2}\frac{(\omega - kv)^2}{[(1 - v^2/c^2)^3]^{1/2}}\left[\cos 2(\omega t - kx) - 2\,\frac{v}{c}\sin 2(\omega t - kx)\right]\Bigg\} \end{aligned} \tag{7}$$

From this formula, an important physical consequence follows; i.e., that

$$\Delta\tau_q = 0 \qquad \text{or} \qquad \langle ds_q\rangle = 0 \tag{8}$$

for the photon, for which $\omega - kv \Rightarrow \omega - kc = 0$ ($v = pc^2/\varepsilon = kc^2/\omega = c$). This means that the velocity of light does not change and its invariant character is still valid in quantum space–time.

2.2. Four-Momentum of a Particle in Quantum Space–Time

By definition, in quantum space–time the four-momentum P^μ is given by

$$P^\mu = mc\,\frac{dz^\mu}{ds_q} = m\,\frac{dz^\mu}{d\tau_q}$$

Taking into account expressions (1) and (4), we have

$$P^i = mv^i(\beta+\delta) + mcL\frac{\partial\Pi^i}{\partial x^\rho}V^\rho\left(\beta - L\beta^3\frac{\partial\Pi^\mu}{\partial x^\rho}V^\rho V^\mu\right)$$
$$P^0 = mc(\beta+\delta) + mcL\frac{\partial\Pi^0}{\partial x^\rho}V^\rho\left(\beta - L\beta^3\frac{\partial\Pi^\mu}{\partial x^\rho}V^\rho V^\mu\right) \qquad (9)$$

The square of these quantities takes the form

$$(P^i)^2 = m^2(v^i)^2\beta^2(1+\varkappa) + m^2c^2L^2\beta^2\left(\frac{\partial\Pi^i}{\partial x^\rho}V^\rho\right)^2 - 3cm^2L^2\beta^4\left(v^j\frac{\partial\Pi^j}{\partial x^\rho}V^\rho\right)$$
$$\times\left(\frac{\partial\Pi^\mu}{\partial x^\rho}V^\rho V^\mu\right) - m^2cL^2\beta^4\left(\frac{\partial\Pi^\mu}{\partial x^\rho}V^\rho V^\mu\right)\left(v^j\frac{\partial\Pi^j}{\partial x^\rho}V^\rho\right)$$
$$+2cm^2L\beta^2\left(v^j\frac{\partial\Pi^j}{\partial x^\rho}V^\rho\right)$$

$$(P^0)^2 = m^2c^2\beta^2(1+\varkappa) + L^2c^2m^2\beta^2\left(\frac{\partial\Pi^0}{\partial x^\rho}V^\rho\right)^2$$
$$+2m^2c^2L\beta^2\frac{\partial\Pi^0}{\partial x^\rho}V^\rho - 3m^2c^2L^2\beta^4\frac{\partial\Pi^0}{\partial x^\rho}V^\rho\left(\frac{\partial\Pi^\mu}{\partial x^\delta}V^\delta V^\mu\right)$$
$$-m^2c^2L^2\beta^4\left(\frac{\partial\Pi^\mu}{\partial x^\rho}V^\rho V^\mu\right)\left(\frac{\partial\Pi^0}{\partial x^\rho}V^\rho\right)$$

or

$$(P^0)^2 = (P^i)^2 + m^2c^2(1+\varkappa) + m^2L^2c^2\beta^2\left(\frac{\partial\Pi^\nu}{\partial x^\rho}V^\rho\frac{\partial\Pi^\nu}{\partial x^\delta}V^\delta\right)$$
$$-4m^2c^2L^2\beta^4\left(V^\nu\frac{\partial\Pi^\nu}{\partial x^\rho}V^\rho\right)\left(\frac{\partial\Pi^\mu}{\partial x^\delta}V^\delta V^\mu\right)$$
$$+2m^2c^2L\beta^2\left(V^\nu\frac{\partial\Pi^0}{\partial x^\rho}V^\rho\right)$$

where

$$\varkappa = -2L\beta^2\frac{\partial\Pi^\mu}{\partial x^\rho}V^\rho V^\mu - L^2\beta^2\left(\frac{\partial\Pi^\mu}{\partial x^\delta}V^\delta\right)^2 + 4L^2\beta^4\left(\frac{\partial\Pi^\mu}{\partial x^\rho}V^\rho V^\mu\right)^2$$

It is important to notice that in quantum space–time the relation

$$(P^0)^2 = (P^i)^2 + m^2c^2 \qquad (P^0 = \varepsilon/c) \qquad (10)$$

between the energy and momentum of the particle is valid for any $\Pi^\mu(x)$, i.e., it is independent of the concrete form of the function $\Pi^\mu(x)$.

For the function (5) averaged values of $\langle(P^0)^2\rangle$ and $\langle(P^x)^2\rangle$ take the form

$$\langle(P^x)^2\rangle = m^2v^2\beta^2(1-L^2\beta^2I_1+4L^2\beta^4I_2)-\tfrac{1}{2}m^2c^2L^2\beta^2I_1 - 4m^2cL^2\beta^4(v/c^2)(\omega-kv)^2\sin 2(\omega t-kx)$$

$$\langle(P^0)^2\rangle = m^2c^2\beta^2(1-L^2\beta^2I_1+4L^2\beta^4I_2)+\tfrac{1}{2}m^2c^2L^2\beta^2I_1 - 2m^2c^2L^2\beta^4I_1+4m^2c^2L^2\beta^4(v/c^3)(\omega-kv)^2\sin 2(\omega t-kx)$$

where I_1 and I_2 are expressed by (6). From these expressions, it is easy to verify that

$$\langle(P^0)^2\rangle = \langle(P^x)^2\rangle + m^2c^2$$

as one would expect. But this relation holds by formula (10) without taking the trace of γ matrices.

It is clear that in accordance with formula (10), the components of the four-velocity depend on each other:

$$U^\mu U^\mu = 1, \qquad U^\mu = P^\mu/mc \tag{11}$$

At the same time, we notice that this equality follows immediately from definition (2) and the formula $ds_q^2 = \eta_{\mu\nu}\, dz^\mu\, dz^\nu$. Geometrically, one can see that in quantum space-time U^μ is also a unit vector. By analogy with the definition of the four-velocity, we call the second derivative

$$\frac{d^2z^\mu}{ds_q^2} = \frac{dU^\mu}{ds_q}$$

the four-acceleration. Differentiating relation (11), we find $U^\mu\, dU^\mu/ds_q = 0$, i.e., in quantum space-time, four-vectors of velocity and acceleration are "mutually perpendicular."

Finally, we consider the case of the Pauli metric for the tetrad formalism, for which $\eta_{ab}\to\delta_{ab}$ in the above formulas should be changed. Thus, the expressions for $\Delta\tau_q$, $\langle(P^x)^2\rangle$, and $\langle(P^0)^2\rangle$ take the form

$$\Delta\tau_q = dt\left[\left(1-\frac{v^2}{c^2}\right)^{1/2} - \frac{L^2}{2c^2}\frac{(\omega-kv)^2}{[(1-v^2/c^2)^3]^{1/2}}\right]$$

$$\langle(P^x)^2\rangle = m^2v^2\beta^2(1+4L^2\beta^4I)+m^2L^2c^2\beta^2I$$

$$\langle(P^0)^2\rangle = m^2c^2\beta^2(1+4L^2\beta^4I)+m^2L^2c^2\beta^2I-4m^2c^2L^2\beta^4I$$

where $I=(\omega-kv)^2c^{-2}$. The essential difference between these formulas and those obtained above is that here oscillating terms are absent.

2.3. Appearance of an Additional Force Due to Quantum Space–Time

Generally speaking, since the concept of force has physical meaning only at the classical physical level, before defining four-force in quantum space-time, we first consider its nonrelativistic limit. To define force in quantum space-time, we proceed as follows. Consider $(P^i)^2$ and obtain its nonrelativistic value, that is,

$$(P^i)^2 = m^2(v^i)^2 + m^2L^2\left(\frac{\partial \Pi^i}{\partial t} + \frac{\partial \Pi^i}{\partial x^j}\, v^j\right)^2$$

where the term proportional to the single matrix $\Pi^i(x)$ (i.e., term of the order of L) is omitted, which, of course, vanishes after taking the trace. Further, according to the correspondence principle, with this quantity one can construct the Lagrangian function of the free particle in quantum space-time,

$$\mathscr{L}(x^i, \dot{x}^i, t) = \frac{(P^i)^2}{2m} = \frac{m(v^i)^2}{2} + \frac{mL^2}{2}\left(\frac{\partial \Pi^i}{\partial t} + \frac{\partial \Pi^i}{\partial x^j}\, v^j\right)^2 \tag{12}$$

With this Lagrangian function, as usual one can reformulate the action principle and obtain the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}^n} - \frac{\partial \mathscr{L}}{\partial x^n} = 0, \qquad n = 1, 2, 3$$

The latter gives an equation of motion for the free particle in space-time over the large scale:

$$m\ddot{x}^n = F_q^n, \qquad n = 1, 2, 3 \tag{13a}$$

where

$$\begin{aligned} F_q^n = -\frac{mL^2}{2}\bigg(& \frac{\partial^2 \Pi^i}{\partial t^2}\frac{\partial \Pi^i}{\partial x^n} + \frac{\partial \Pi^i}{\partial x^n}\frac{\partial^2 \Pi^i}{\partial t^2} + \frac{\partial \Pi^i}{\partial x^n}\frac{\partial^2 \Pi^i}{\partial t\, \partial x^k}\, v^k + \frac{\partial \Pi^i}{\partial x^n}\frac{\partial \Pi^i}{\partial x^k}\, \dot{v}^k \\ & + \frac{\partial^2 \Pi^i}{\partial x^j\, \partial t}\, v^j \frac{\partial \Pi^i}{\partial x^n} + \frac{\partial \Pi^i}{\partial x^j}\, \dot{v}^j \frac{\partial \Pi^i}{\partial x^n} - \frac{\partial \Pi^i}{\partial t}\frac{\partial^2 \Pi^i}{\partial x^j\, \partial x^n}\, v^j - \frac{\partial^2 \Pi^i}{\partial x^j\, \partial x^n}\, v^j \frac{\partial \Pi^i}{\partial t} \\ & - \frac{\partial^2 \Pi^i}{\partial x^j\, \partial x^n}\frac{\partial \Pi^i}{\partial x^k}\, v^j v^k - \frac{\partial \Pi^i}{\partial x^j}\frac{\partial^2 \Pi^i}{\partial x^k\, \partial x^n}\, v^j v^k \bigg) \end{aligned} \tag{13b}$$

After taking the trace, this force is reduced to the following averaged value (for Feynman's metric in the tetrad formalism):

$$\begin{aligned} \langle F_q^n \rangle = -mL^2\eta_{ab}\bigg(& \frac{\partial^2 e_a^i}{\partial t^2}\frac{\partial e_b^i}{\partial x^n} + \frac{\partial e_a^i}{\partial x^n}\frac{\partial^2 e_b^i}{\partial t\, \partial x^k}\, v^k + \frac{\partial e_a^i}{\partial x^n}\frac{\partial e_b^i}{\partial x^k}\, \dot{v}^k \\ & - \frac{\partial e_a^i}{\partial t}\frac{\partial^2 e_b^i}{\partial x^j\, \partial x^n}\, v^j - \frac{\partial e_a^i}{\partial x^k}\frac{\partial^2 e_b^i}{\partial x^n\, \partial x^j}\, v^j v^k \bigg) \end{aligned}$$

In two-dimensional space-time, when $e_a^i(x)$ is given by (5) {here it should be assumed that x depends on time variable t, i.e., for example, $\cos(\omega t - kx) \equiv \cos[\omega t - kx(t)]$}, this averaged force takes the form

$$\langle F_q^x \rangle = mL^2[(k\omega^2 - 2k^2\omega\dot{x}) \sin 2\delta + 2k^2\ddot{x} \cos 2\delta], \qquad \delta = \omega t - kx \tag{14a}$$

and

$$\langle F_q^x \rangle = -2mL^2k^2\ddot{x} \tag{14b}$$

for the Feynman and Pauli metrics, respectively.

Thus, we see that due to quantum space-time at small distances, in our model free particle motion is changed and an additional force appears. This force, depending on the velocity and acceleration of the particle, either oscillates or has a friction force character in accordance with the choice of the Feynman or the Pauli metric for tetrad space-time, respectively. In the oscillating force case, it seems that particle motion, whether slowed down or accelerated, moves periodically by means of chain steps. In contrast, in the friction force case [second term in (14)], the particle undergoes a purely slowing-down motion.

In Section 4, we will study the motion of a particle given by an equation of the type of (13). Now we consider the case when the matrix function $\Pi^i(x)$, $i = 1, 2, 3$, does not depend on the time variable explicitly. For this stationary case, the Lagrangian (12) has the following structure:

$$\mathscr{L}(x^i, \dot{x}^i) = \frac{m(v^i)^2}{2} + \frac{mL^2}{2}\left(\frac{\partial \Pi^i}{\partial x^j} v^j\right)^2 \tag{15a}$$

and therefore the force F_q^n in equation (13) is given by

$$F_q^n = -\frac{mL^2}{2}\left(\frac{\partial \Pi^i}{\partial x^n}\frac{\partial^2 \Pi^i}{\partial x^j\,\partial x^m} v^j v^m + \frac{\partial^2 \Pi^i}{\partial x^j\,\partial x^m} v^j v^m \frac{\partial \Pi^i}{\partial x^n} + \frac{\partial \Pi^i}{\partial x^n}\frac{\partial \Pi^i}{\partial x^j}\dot{v}^j + \frac{\partial \Pi^i}{\partial x^j}\frac{\partial \Pi^i}{\partial x^n}\dot{v}^j\right) \tag{15b}$$

or

$$\langle F_q^n \rangle = -mL^2\eta_{ab}\left(\frac{\partial^2 e_a^i}{\partial x^j\,\partial x^m} v^j v^m + \frac{\partial e_a^i}{\partial x^j}\dot{v}^j\right)\frac{\partial e_b^i}{\partial x^n}$$

Here, we use another representation for the e_a^i field, and choose the spherical frame of reference as the tetrad coordinate system and the Cartesian one for the world coordinate system (Dineykhan and Namsrai, 1985a). Then

$$e_a^i(x) = \begin{pmatrix} x/r & y/r & z/r \\ zx/r\rho & zy/r\rho & -\rho/r \\ -y/\rho & x/\rho & 0 \end{pmatrix} \tag{16}$$

where $r = (x^2 + y^2 + z^2)^{1/2}$ and $\rho = (x^2 + y^2)^{1/2}$. To define the motion of the particle it is assumed that the variables x, y, z and therefore r and ρ depend on the time variable t. With (16), the Lagrangian (15a) has the form

$$\mathscr{L}=\frac{m(v^i)^2}{2}+\frac{L^2}{mr^4}(M^i)^2+\frac{L^2}{m\rho^4}\left(\frac{z}{r}\right)^2 M_z^2 \tag{17}$$

where

$$M^i=\varepsilon^{ijk}r^j p^k, \qquad M_z=xp_y-yp_x$$

are the angular momentum and its third component, respectively. The last term in (17) arises from a peculiarity of the coordinate transformation only and corresponds to a string ("Dirac veto")-like singularity, which is ruled out by another choice of the tetrad field, say

$$[e_a^i(x)]_0=(z/r)\begin{pmatrix} x/\rho & y/\rho & 0 \\ -y/\rho & x/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The latter yields only the last term in (17). Thus, in our scheme, by analogy with quantum field theory, there exists some subtractive procedure making it possible to obtain a finite value in quantum space–time. For the nonrelativistic limit, when $\Pi^i(x)$ does not depend on time, the Galilean-invariant Lagrangian function of a free particle acquires the form

$$\mathscr{L}(x)=\frac{(p^i)^2}{2m}+\frac{L^2}{r^4}\frac{(M^i)^2}{m} \tag{18}$$

In a previous paper (Dineykhan and Namsrai, 1985b) we investigated particle motion by this Lagrangian function and showed that, depending on initial conditions, a particle's trajectory is complicated and the particle makes a spiral-like motion along the direction of the rectilinear classical trajectory. In Section 4, the Lagrangian (18) will be needed to discuss the possibility of occurrence of anisotropy of inertia due to quantum space–time.

By analogy with the usual definition of force, we now define its four-vector as the derivative

$$F^\mu=\frac{dP^\mu}{ds_q}=mc\frac{dU^\mu}{ds_q} \tag{19}$$

Since $U^\mu\, dU^\mu/ds_q=0$, components of the four-force also satisfy the same identity $F^\mu U^\mu=0$. We calculate the explicit form of (19) by using the definition of U^μ and ds_q in quantum space–time. Thus,

$$F^\mu=m\frac{d^2z^\mu}{c\,d\tau_q^2}=m\frac{d^2x^\mu}{c\,d\tau_q^2}+\frac{mL}{c}\frac{\partial^2\Pi^\mu}{\partial x^\rho\,\partial x^\delta}\frac{dx^\delta}{d\tau_q}\frac{dx^\rho}{d\tau_q}$$
$$+\frac{mL}{c}\frac{\partial\Pi^\mu}{\partial x^\rho}\frac{d^2x^\rho}{d\tau_q^2} \tag{20}$$

where the following simple connections exist between derivatives with respect to variables $d\tau_q$ and $d\tau$:

$$\frac{dx^\rho}{c\,d\tau_q}=\frac{dx^\rho}{c\,d\tau}\frac{d\tau}{d\tau_q}$$

$$\frac{d^2x^\mu}{c\,d\tau_q^2}=\frac{d^2x^\mu}{c\,d\tau^2}\left(\frac{d\tau}{d\tau_q}\right)^2+\frac{dx^\mu}{c\,d\tau}\frac{d}{d\tau_q}\left(\frac{d\tau_q}{d\tau}\right)^{-1}$$

$$=\frac{d^2x^\mu}{c\,d\tau^2}\left(\frac{d\tau_q}{d\tau}\right)^{-2}-\frac{dx^\mu}{c\,d\tau}\left(\frac{d\tau_q}{d\tau}\right)^{-2}\frac{d^2\tau_q}{d\tau^2}\left(\frac{d\tau_q}{d\tau}\right)^{-1}$$

Here

$$d\tau=dt\,(1-v^2/c^2)^{1/2}$$

In accordance with the definitions

$$\begin{aligned} u^\alpha&=\frac{dx^\alpha}{c\,d\tau}=\left\{\left(1-\frac{v^2}{c^2}\right)^{-1/2},\frac{\mathbf{v}}{c}\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right\}\\ f^\mu&=m\frac{d^2x^\mu}{c\,d\tau^2}=\left\{\frac{\mathbf{f}}{c}\left(1-\frac{v^2}{c^2}\right)^{-1/2},\frac{\mathbf{f}\cdot\mathbf{v}}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right\}\end{aligned}\tag{21}$$

for the usual relativistic mechanics (in the case when $L=0$, here $\mathbf{f}$ is the classical force), we rewrite expression (20) in the form

$$\begin{aligned}F^\mu=&\left(f^\mu+mcL\frac{\partial^2\Pi^\mu}{\partial x^\rho\,\partial x^\delta}u^\delta u^\rho+L\frac{\partial\Pi^\mu}{\partial x^\rho}f^\rho\right)\left(\frac{d\tau_q}{d\tau}\right)^{-2}\\ &-\left(mu^\mu+mL\frac{\partial\Pi^\mu}{\partial x^\rho}u^\rho\right)\left(\frac{d\tau_q}{d\tau}\right)^{-2}\frac{d^2\tau_q}{d\tau^2}\left(\frac{d\tau_q}{d\tau}\right)^{-1}\end{aligned}\tag{22}$$

Here the connection between the variables $d\tau_q$ and $d\tau$ is given by

$$d\tau_q/d\tau=(1+2LN+L^2M)^{1/2}$$

$$N=\frac{\partial\Pi^\mu}{\partial x^\rho}u^\rho u^\mu,\qquad M=\left(\frac{\partial\Pi^\mu}{\partial x^\rho}u^\rho\right)^2$$

and therefore

$$\begin{aligned}(d\tau_q/d\tau)^{-1}&=1-\tfrac{1}{2}(2LN+L^2M)+\tfrac{3}{2}L^2N^2\\ (d\tau_q/d\tau)^{-2}&=1-2LN-L^2M+4L^2N^2\\ d^2\tau_q/d\tau^2&=\tfrac{1}{2}(1+2LN+L^2M)^{-1/2}(2LA+L^2B)\\ &=\tfrac{1}{2}(2LA+L^2B-2L^2NA)\end{aligned}\tag{23}$$

with

$$A = c\left[\frac{\partial^2\Pi^\mu}{\partial x^\rho\,\partial x^\delta}\,u^\rho u^\delta u^\mu + \frac{\partial\Pi^\mu}{\partial x^\rho}\,(w^\rho u^\mu + u^\rho w^\mu)\right]$$

$$B = c\left[\left(\frac{\partial^2\Pi^\mu}{\partial x^\rho\,\partial x^\delta}\,\frac{\partial\Pi^\mu}{\partial x^n} + \frac{\partial\Pi^\mu}{\partial x^\rho}\,\frac{\partial^2\Pi^\mu}{\partial x^\delta\,\partial x^n}\right)u^\rho u^\delta u^n + \frac{\partial\Pi^\mu}{\partial x^\rho}\,\frac{\partial\Pi^\mu}{\partial x^\delta}\,(w^\rho u^\delta + u^\rho w^\delta)\right] \tag{24}$$

$$w^\rho = du^\rho / c\,d\tau \equiv f^\rho / mc$$

Now we define the averaged force over the large scale. For this, taking into account expression (22) and inserting (23) and (24) into it, and carrying out a trace of γ^a matrices, we get

$$\langle f^\mu \rangle = f^\mu + L^2 Q^\mu \tag{25}$$

where

$$Q^\mu = \tfrac{1}{4}\,Sp\,I^\mu$$

$$I^\mu = f^\mu(4N^2 - M) - 2X^\mu N - mu^\mu(\tfrac{1}{2}B - 3NA - AN) - mY^\mu A$$

Here

$$X^\mu = mc\,\frac{\partial^2\Pi^\mu}{\partial x^\delta\,\partial x^\rho}\,u^\rho u^\delta + \frac{\partial\Pi^\mu}{\partial x^\rho}\,f^\rho, \qquad Y^\mu = \frac{\partial\Pi^\mu}{\partial x^\rho}\,u^\rho$$

Finally, one can take the trace of expression (25) for the force in quantum space–time by using the tetrad representation, as done above, but in the given case it is not so important to obtain the explicit form of (25).

2.4. Dynamical Equation and Quantum Mechanical Consideration of Particle Motion in Quantum Space–Time

Now, when one knows how to calculate F^μ over the large scale, one can use the differential equation (19) in order to find four connected variables $x^\mu(\tau)$ and then τ is eliminated to obtain $\mathbf{x}(t)$. This procedure is valid in our case, since in quantum space–time the condition $u^\mu u^\mu = U^\mu U^\mu = 1$ is always invariant. This relation guarantees that initial values of u^μ should be chosen so that $d\tau$ is indeed the proper time.

Our main assertion is that even when an external force is absent, the particle dynamics is determined by equation (19), where in the capacity of force $\mathbf{f}$ in (21) now should be put the quantum force given by equalities (13b), (14), or (15b). In usual space–time on a large scale this is equivalent to the following equation:

$$mc\,d^2x^\beta / d\tau^2 = f^\beta \tag{26}$$

where f^{β} is given by (21), where the force **f** is in turn defined by (13b) or (15b).

Thus, we see that due to the quantum nature of space-time, the dynamics of the particle differs from Newtonian dynamics by an additional force f_q^i:

$$m\ddot{x}^i = f_{\rm ex}^i + f_q^i \tag{27}$$

where $f_{\rm ex}^i$ and f_q^i are external and internal forces. The latter is caused by quantum space-time and is given by formula (13b) or (15b), depending on the concrete representation of the tetrad field. We notice that even in the absence of an external force a particle does not move along the rectilinear trajectory. However, at the classical physical level the additional force (13b) or (15b) is very small, and as a result a particle undergoes almost rectilinear motion (for details, see Dineykhan and Namsrai, 1985b). In Section 4, we will study equation (27) in a gravitational force field.

Now we consider the nonrelativistic quantum mechanical method from the quantum space-time point of view. Here we briefly discuss only the stationary problem of electron motion in hydrogenlike atoms by means of the Lagrangian function (18). According to the correspondence principle, in the quantum mechanical case the operator-valued effective Hamiltonian for a particle moving in the external potential field $U(r)$ consists of two parts:

$$\hat{H} = \hat{H}_0 + \delta\hat{H} \tag{28}$$

where

$$\hat{H}_0 = \hat{\mathbf{p}}^2/2m + U(r)$$

and

$$\delta\hat{H} = (L^2/r^4)\hbar^2\hat{l}^2/m \tag{29}$$

in accordance with (18). Here we carry out the usual substitution $\mathbf{M}^2 \to \hbar^2\hat{l}^2$, where $\hat{l}$ is the operator of the angular momentum. Further, we are only interested in the calculation of the contribution to the energy level of the electron due to the additional interaction (29) caused by quantum space-time. It is obvious that by this interaction the electron energy level undergoes an additional shift ΔE_q, which may be estimated by using the nonrelativistic wave function $\psi(\mathbf{r})$ of the electron. After a standard calculation, we find

$$\Delta E_q = \frac{1}{2n^5}\frac{Z^4L^2\hbar^2}{(l+\frac{1}{2})(l-\frac{1}{2})(l+\frac{3}{2})}\left(\frac{m_e e^2}{\hbar^2}\right)^2\frac{1}{m_e}[3n^2 - l(l+1)] \tag{30}$$

where m_e is the electron mass. This expression will be used in Section 4 to obtain bounds on the value of the fundamental length.

3. QUANTIZATION OF GRAVITY (PRELIMINARY RESULTS)

3.1. Reformulation of the Equivalence Principle

It is well known that the equivalence principle between gravity and inertia can be understood as a reaction of a physical system on the external gravitational field. It is asserted that no external static homogeneous gravitational field whatever can be detected in a freely falling elevator, since in this field an observer, test body, and the elevator itself acquire the same acceleration. Following Weinberg (1972), one can easily prove this for an N-particle system moving with nonrelativistic velocity under an action force (for example, electromagnetic and gravitational) $\mathbf{f}(\mathbf{x}_n - \mathbf{x}_m)$ in the external gravitational field. The equation of motion is

$$m_n\, d^2\mathbf{x}_n/dt^2 = m_n\mathbf{g} + \sum_m \mathbf{f}(\mathbf{x}_n - x_m), \qquad n, m = 1, 2, \ldots, N \tag{31}$$

Assuming the following non-Galilean transformation of space–time coordinates

$$\mathbf{x}' \to \mathbf{x} - \tfrac{1}{2}\mathbf{g}t^2, \qquad t' = t \tag{32}$$

one finds that the term with $\mathbf{g}$ is compensated by the inertial "force" and the motion equation takes the form

$$m_n\, d^2\mathbf{x}'_n/dt'^2 = \sum_m \mathbf{f}(\mathbf{x}'_n - \mathbf{x}'_m) \tag{33}$$

Therefore, an observer O using coordinates $\mathbf{x}$, t and a freely falling colleague O' using coordinates $\mathbf{x}'$, t' do not find any difference in the laws of mechanics, with the exception that O will observe the influence of a gravitational field, but O' will not.

It is easily verified that due to the additional force given by (13b) or (15b) in space–time on a large scale, continued from the quantum one, the equivalence principle formulated by the above formula is not valid. Indeed, equations (31) and (33) take the form

$$m_n \frac{d^2\mathbf{x}_n}{dt^2} = \mathbf{f}_q(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, t) + m_n\mathbf{g} + \sum_m \mathbf{f}(\mathbf{x}_n - \mathbf{x}_m) \tag{34}$$

and

$$m_n \frac{d^2\mathbf{x}'_n}{dt'^2} = \mathbf{f}'_q(\mathbf{x}', \dot{\mathbf{x}}', \ddot{\mathbf{x}}', t', \mathbf{g}) + \sum_m \mathbf{f}(\mathbf{x}'_n - \mathbf{x}'_m) \tag{35}$$

In accordance with transformation (32), the force $f'_q(\mathbf{x}', \dot{\mathbf{x}}', \ddot{\mathbf{x}}', t', \mathbf{g})$ depending on $\mathbf{g}$ has quite a different structure than the force $\mathbf{f}_q(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, t)$. Therefore, in contrast to the standard formulation of the equivalence principle, in our case complete compensation between inertial and gravitational forces is not achieved and both observers O and O' should detect an influence of the gravitational field. However, for the second observer O' this influence is an infinitesimally small value, of the order of $O(L^2)$.

Thus, for a static homogeneous gravitational field, the equivalence principle is slightly violated in quantum space-time and the difference in mechanical laws for observers O and O' is of the order of $O(L^2)$. For classical physical phenomena this may be excluded from consideration. However, as will be shown in Section 4, for processes taking place in the microworld, where the quantum force $\mathbf{f}_q$ in (13b) depends on the particle mass and energy by the formulas $\omega = \varepsilon/\hbar$ and $\kappa = p/\hbar = mv/\hbar$, this difference is significant and gives an antigravitational effect—a friction fifth force at least for quasiclassical particles.

An analogous situation holds when inertial forces do not completely compensate gravitational ones for systems freely falling in a non-homogeneous or time-dependent gravitational field (as in the above case); one can reformulate the equivalence principle in quantum space-time by the assertion that at every point of space-time on a large scale and in an arbitrary gravitational field one can choose a "quasilocal inertial" coordinate system such that in a sufficiently small neighborhood of the given point the laws of nature [given by the special theory of relativity reformulated above, in particular, equations of the type of (13a)] will have the same form up to the order of $O(L^2)$ as in an unaccelerated Cartesian coordinate systems.

Summarizing, we note that the introduction of the hypothesis about the quantum nature of space-time at small distances leads to non-homogeneous and anisotropic space-time on a large scale. However, these structural changes differ from homogeneity and isotropy by the order of $O(L^2)$. Thus, it is natural that the equivalence principle is only approximately valid and may be formulated only at this level of accuracy.

3.2. Gravitational Forces in Quantum Space–Time

Consider a "freely" moving particle under the action of purely gravitational forces. Here the word "freely" means that an additional force caused by the quantum nature of space-time also acts on the particle. According to the equivalence principle reformulated above [we call it the "slightly violated principle of equivalence"] there exists a "freely" falling system of reference ξ^α in which a particle moves along an almost rectilinear trajectory given by the equation

$$\frac{d^2\xi^\alpha}{d\tau^2} = \frac{1}{m} f^\alpha(\xi) \tag{36a}$$

where the averaged force f^α is proportional to the value of L^2 and is given by expression (21); here **f** is in turn determined by (13b) or (15b). The $d\tau$, in (36a) is the proper time

$$d\tau^2 = \eta_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta \tag{36b}$$

[compare this with equations (3) and (26)]. Now take a curvilinear quantum system of reference z^μ connected with the usual curvilinear one x^μ by relation (1). In this case, the coordinates ξ^α of the freely falling system of reference are functions of z^μ and x^μ; then equation (36a) acquires the form

$$\frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial z^\mu}\frac{dz^\mu}{d\tau}\right) = \frac{\partial\xi^\alpha}{\partial z^\mu}\frac{d^2 z^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial z^\mu\,\partial z^\nu}\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau} = \frac{1}{m} f^\alpha(\xi) \tag{37a}$$

Here one should define an inverse operation of $\partial z^\mu/\partial x^\alpha = \delta^\mu_\alpha + L\,\partial\Pi^\mu/\partial x^\alpha$. What do we mean by $\partial/\partial z^\mu$? For an explanation of this, we multiply this equation from the left-hand side by $\partial z^\lambda/\partial\xi^\alpha$ and get

$$\frac{\partial z^\lambda}{\partial\xi^\alpha}\frac{\partial\xi^\alpha}{\partial z^\mu}\frac{d^2 z^\mu}{d\tau^2} + \frac{\partial z^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial z^\mu\,\partial z^\nu}\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau} = \frac{1}{m}\frac{\partial z^\lambda}{\partial\xi^\alpha} f^\alpha(\xi) \tag{37b}$$

It is obvious that in our case the well-known multiplication rule

$$\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial\xi^\alpha} = \delta^\lambda_\mu \tag{38}$$

for c-number variables ξ^α and x^μ does not work.

A generalization of the definition (38) is crucial for noncommuting variable z^μ in the quantum space–time case. Furthermore, because of noncommutability of z^μ, an operation of the type of (38) is not commutative,

$$\frac{\partial\xi^\alpha}{\partial z^\mu}\frac{\partial z^\lambda}{\partial\xi^\alpha} = \frac{\partial\xi^\alpha}{\partial x^\rho}\frac{\partial x^\rho}{\partial z^\mu}\frac{\partial z^\lambda}{\partial x^\delta}\frac{\partial x^\delta}{\partial\xi^\alpha} = \frac{\partial x^\rho}{\partial z^\mu}\frac{\partial z^\lambda}{\partial x^\rho} \neq \frac{\partial z^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial z^\mu} \tag{39}$$

Here we have used definition (38). Noncommutability of these Jacobians has a deeper meaning and is caused by the nonequivalence properties of the usual and quantum space–time with respect to the transformation law leading to the passage from one to the other. Indeed, $\partial x^\rho/\partial z^\mu$ corresponds to the Jacobian of transformation from quantum space-time coordinates z^μ to the usual ones x^ρ, and vice versa $\partial z^\lambda/\partial x^\rho$ means that the passage is carried out from x^ρ to z^λ points. For simplicity, let indices $\mu = \lambda = \delta$ coincide; then the transformation corresponding to (39) may be illustrated by means of Figure 1. Thus, there are two possibilities:

$$\frac{\partial x^\rho}{\partial z^\mu}\frac{\partial z^\lambda}{\partial x^\rho} = \delta^\lambda_\mu, \qquad \frac{\partial z^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial z^\mu} \neq \delta^\lambda_\mu \tag{40}$$

or

$$\frac{\partial x^\rho}{\partial z^\mu}\frac{\partial z^\lambda}{\partial x^\rho} \neq \delta^\lambda_\mu, \qquad \frac{\partial z^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial z^\mu} = \delta^\lambda_\mu \tag{41}$$

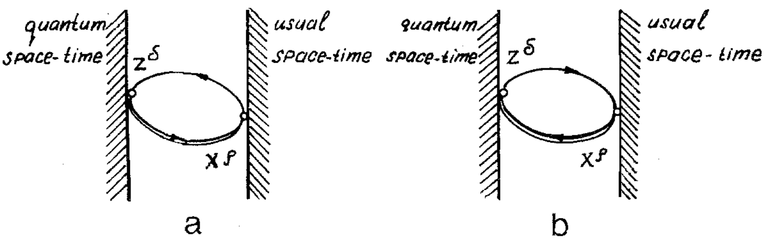

Fig. 1. Illustration of the nonequivalence of two cycle transformations: (a) The passage from quantum space-time point z^δ and the return to it through the usual point x^ρ, and vice versa; (b) the transformation starting from the usual space-time point x^ρ and the return to it through the "quantum" point z^δ. As a result, the points move along closed lines; here doubled lines correspond to the initial positions of moving points.

The first term in (40) means the strict return of point z^λ in quantum space-time to its initial position after an "excursion" over usual space-time (through its point x^ρ) (Figure 1a). In turn this means that the latter does not possess any strange properties leading to different results with respect to the coordinate transformation. For usual space-time this should be so. But it is optional to expect the same rule for quantum space-time [second term in (40)]. Strictly speaking, in this case the strict return of the usual space-time point x^ρ to its initial position after "excursion" over quantum space-time (through its point z^δ) is impossible, due to the quantum nature of the latter (Figure 1b). Further, for case (41) there appear situations contrary to the ones discussed above.

Thus, the question is how to choose between the two possibilities (40) and (41). Here we prefer to use rule (40). Our next problem is to define the operation $\partial x^\rho/\partial z^\mu$ in accordance with this rule. One can verify that the definition

$$\frac{\partial x^\rho}{\partial z^\mu}=\delta^\rho_\mu - L\frac{\partial \Pi^\rho}{\partial x^\mu}+L^2\frac{\partial \Pi^\alpha}{\partial x^\mu}\frac{\partial \Pi^\rho}{\partial x^\alpha}-L^3\frac{\partial \Pi^\alpha}{\partial x^\mu}\frac{\partial \Pi^\beta}{\partial x^\alpha}\frac{\partial \Pi^\rho}{\partial x^\beta}+\cdots \tag{42}$$

satisfies the first condition in (40). With this definition, we have a series of equalities,

$$\begin{aligned}
&\frac{\partial x^\rho}{\partial z^\mu}\frac{\partial z^\lambda}{\partial x^\rho}=\delta^\lambda_\mu\\
&\frac{\partial z^\lambda}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial z^\mu}=\delta^\lambda_\mu+L^2\left(\frac{\partial \Pi^\rho}{\partial x^\mu}\frac{\partial \Pi^\lambda}{\partial x^\rho}-\frac{\partial \Pi^\lambda}{\partial x^\rho}\frac{\partial \Pi^\rho}{\partial x^\mu}\right)\\
&\frac{\partial x^\lambda}{\partial z^\alpha}\frac{\partial z^\alpha}{\partial x^\mu}=\delta^\lambda_\mu+L^2\left(\frac{\partial \Pi^\rho}{\partial x^\mu}\frac{\partial \Pi^\lambda}{\partial x^\rho}-\frac{\partial \Pi^\lambda}{\partial x^\rho}\frac{\partial \Pi^\rho}{\partial x^\mu}\right)\\
&\frac{\partial z^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial z^\alpha}=\delta^\lambda_\mu
\end{aligned} \tag{43}$$

If we use the tetrad formalism $\Pi^\rho(x)=\gamma^a e_a^\rho(x)$, then the second term in the second and third equations of (43) acquires the form

$$L^2\left(\frac{\partial\Pi^\rho}{\partial x^\mu}\frac{\partial\Pi^\lambda}{\partial x^\rho}-\frac{\partial\Pi^\lambda}{\partial x^\rho}\frac{\partial\Pi^\rho}{\partial x^\mu}\right)=2i\sigma_{ab}\frac{\partial e_a^\rho}{\partial x^\mu}\frac{\partial e_b^\lambda}{\partial x^\rho}L^2 \tag{44}$$

where

$$\sigma_{ab}=\frac{1}{2i}(\gamma^a\gamma^b-\gamma^b\gamma^a)$$

It is important to notice that the difference in transformations (40) and (41) is proportional to the L^2 term, i.e., the commutator between them is given by

$$\left[\frac{\partial z^\lambda}{\partial x^\rho},\frac{\partial x^\rho}{\partial z^\mu}\right]_-=2iL^2\sigma_{ab}\frac{\partial e_a^\rho}{\partial x^\mu}\frac{\partial e_b^\lambda}{\partial x^\rho} \tag{45}$$

Thus, in the quantum space–time case transformation Jacobians have quantum nature, whose commutator is given by a relation of the type (45).

Now we return to equation (37b). Taking into account expression (43), we have

$$\frac{d^2z^\lambda}{d\tau^2}+2iL^2\sigma_{ab}\frac{\partial e_a^\rho}{\partial x^\mu}\frac{\partial e_b^\lambda}{\partial x^\rho}\frac{d^2z^\mu}{d\tau^2}+\Gamma^\lambda_{\nu\mu}(z)\frac{dz^\nu}{d\tau}\frac{dz^\mu}{d\tau}=\frac{1}{m}f^\lambda(x) \tag{46}$$

On the right-hand side of this equation we have used the following approximation:

$$\begin{aligned}\frac{\partial z^\lambda}{\partial\xi^\alpha}f^\alpha(\xi)&=\frac{\partial z^\lambda}{\partial x^\delta}\frac{\partial x^\delta}{\partial\xi^\alpha}f^\alpha(\xi)\\&=\left(\delta^\lambda_\delta+L\frac{\partial\Pi^\lambda}{\partial x^\delta}\right)\frac{\partial x^\delta}{\partial\xi^\alpha}f^\alpha(\xi)\approx\frac{\partial x^\lambda}{\partial\xi^\alpha}f^\alpha(\xi)\equiv f^\lambda(x)\end{aligned}$$

since the force $f^\alpha(\xi)$ is proportional to the L^2 term. In equation (46), $\Gamma^\lambda_{\nu\mu}(z)$ is the affine connection for the quantum space–time case, determined by the following formula:

$$\Gamma^\lambda_{\nu\mu}(z)=\frac{\partial z^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial z^\nu\,\partial z^\mu} \tag{47}$$

It is obvious that due to the noncommutability of z^ν variables, the symmetric property of $\Gamma^\lambda_{\nu\mu}(z)$ over indices ν and μ is violated in our case. We separate symmetric and antisymmetric parts in $\Gamma^\lambda_{\nu\mu}(z)$:

$$\Gamma^\lambda_{\nu\mu}(z)=\Gamma^\lambda_{s,\nu\mu}(z)+\Gamma^\lambda_{a,\nu\mu}(z)$$

where

$$\Gamma^\lambda_{s,\nu\mu}(z)=\frac{1}{2}\left(\frac{\partial z^\lambda}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial z^\nu\,\partial z^\mu}+\frac{\partial z^\lambda}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial z^\mu\,\partial z^\nu}\right)$$
$$\Gamma^\lambda_{a,\nu\mu}(z)=\frac{1}{2}\left(\frac{\partial z^\lambda}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial z^\nu\,\partial z^\mu}-\frac{\partial z^\lambda}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial z^\mu\,\partial z^\nu}\right) \tag{48}$$

Now we calculate (47) and (48) and use the usual affine connection

$$\begin{Bmatrix}\lambda\\ \nu\mu\end{Bmatrix}=\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\nu\,\partial x^\mu}$$

For this purpose, we notice that

$$\frac{\partial z^\lambda}{\partial \xi^\alpha}=\frac{\partial z^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial \xi^\alpha}\qquad \frac{\partial}{\partial z^\nu}f(z)=\frac{\partial x^\sigma}{\partial z^\nu}\frac{\partial}{\partial x^\sigma}f(z)$$

Making use of these definitions, we obtain the following expression for (47):

$$\Gamma^\lambda_{\nu\mu}(z)=\begin{Bmatrix}\rho\\ \sigma\delta\end{Bmatrix}\frac{\partial z^\lambda}{\partial x^\rho}\frac{\partial x^\sigma}{\partial z^\nu}\frac{\partial x^\delta}{\partial z^\mu}+\frac{\partial z^\lambda}{\partial x^\delta}\frac{\partial^2 x^\delta}{\partial z^\nu\,\partial z^\mu}$$

Further, in order to estimate this equality, the inverse operation (42) should be used. As a result, we have

$$\Gamma^\lambda_{s,\nu\mu}(z)=\begin{Bmatrix}\lambda\\ \nu\mu\end{Bmatrix}+LQ^\lambda_{\nu\mu}+L^2P^\lambda_{\nu\mu} \tag{49}$$

$$\Gamma^\lambda_{a,\nu\mu}(z)=\tfrac{1}{2}L^2\begin{Bmatrix}\lambda\\ \sigma\delta\end{Bmatrix}\left(\frac{\partial\Pi^\sigma}{\partial x^\nu}\frac{\partial\Pi^\delta}{\partial x^\mu}-\frac{\partial\Pi^\delta}{\partial x^\mu}\frac{\partial\Pi^\sigma}{\partial x^\nu}\right) \tag{50}$$

where

$$Q^\lambda_{\nu\mu}=\begin{Bmatrix}\rho\\ \nu\mu\end{Bmatrix}\frac{\partial\Pi^\lambda}{\partial x^\rho}-\begin{Bmatrix}\lambda\\ \sigma\mu\end{Bmatrix}\frac{\partial\Pi^\sigma}{\partial x^\nu}-\begin{Bmatrix}\lambda\\ \nu\delta\end{Bmatrix}\frac{\partial\Pi^\delta}{\partial x^\mu}-\frac{\partial^2\Pi^\lambda}{\partial x^\mu\,\partial x^\nu}$$

$$P^\lambda_{\nu\mu}=\frac{\partial^2\Pi^n}{\partial x^\nu\,\partial x^\mu}\frac{\partial\Pi^\lambda}{\partial x^n}-\frac{\partial\Pi^\lambda}{\partial x^\delta}\frac{\partial^2\Pi^\delta}{\partial x^\mu\,\partial x^\nu}+f^\lambda_{\nu\mu}+f^\lambda_{\mu\nu}$$

$$f^\lambda_{\nu\mu}=\frac{1}{2}\begin{Bmatrix}\lambda\\ \sigma\delta\end{Bmatrix}\frac{\partial\Pi^\sigma}{\partial x^\nu}\frac{\partial\Pi^\delta}{\partial x^\mu}+\begin{Bmatrix}\lambda\\ \sigma\mu\end{Bmatrix}\frac{\partial\Pi^m}{\partial x^\nu}\frac{\partial\Pi^\sigma}{\partial x^m}$$
$$-\begin{Bmatrix}\rho\\ \sigma\mu\end{Bmatrix}\frac{\partial\Pi^\lambda}{\partial x^\rho}\frac{\partial\Pi^\sigma}{\partial x^\nu}+\frac{\partial\Pi^\rho}{\partial x^\nu}\frac{\partial^2\Pi^\lambda}{\partial x^\mu\,\partial x^\rho}$$

From equation (14) we see that the additional force caused by quantum space–time disappears for the photon or neutrino. Moreover, their proper time (36b) is not independent, since for these particles the right-hand side of (7) or (36b) vanishes. Instead of τ, one uses $\sigma \equiv \xi^0$, so that equations (36a) and (36b) take the form

$$\frac{d^2\xi^\alpha}{d\sigma^2}=0, \qquad \eta_{\alpha\beta}\frac{d\xi^\alpha}{d\sigma}\frac{d\xi^\beta}{d\sigma}=0$$

as in the usual case. By the same method as above, the motion equation in a quantum coordinate system in an arbitrary gravitational field reads

$$\frac{d^2z^\lambda}{d\sigma^2}+2iL^2\sigma_{ab}\frac{\partial e^\rho_a}{\partial x^\mu}\frac{\partial e^\lambda_b}{\partial x^\rho}\frac{d^2z^\mu}{d\sigma^2}+\Gamma^\lambda_{\nu\mu}(z)\frac{dz^\nu}{d\sigma}\frac{dz^\mu}{d\sigma}=0 \tag{51}$$

where $\Gamma^\lambda_{\nu\mu}(z)$ is expressed by the same formula (47).

3.3. Proper Time in the Quantum System of Reference

The proper time (36b) can also be written in an arbitrary (quantum) system of reference:

$$d\tau_q^2=\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial z^\mu}dz^\mu\frac{\partial\xi^\beta}{\partial z^\nu}dz^\nu \tag{52}$$

or

$$d\tau_q^2=\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^n}\frac{\partial x^n}{\partial z^\mu}dz^\mu\frac{\partial\xi^\beta}{\partial x^m}\frac{\partial x^m}{\partial z^\nu}dz^\nu$$

Since, by definition,

$$g_{nm}(x)=\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^n}\frac{\partial\xi^\beta}{\partial x^m}$$

therefore

$$d\tau_q^2=g_{nm}(x)\frac{\partial x^n}{\partial z^\mu}dz^\mu\frac{\partial x^m}{\partial z^\nu}dz^\nu \tag{53}$$

Now the commutator $[dz^\mu, \partial x^m/\partial z^\nu]$ must be defined. Making use of (42), we find in the tetrad formalism

$$\left[dz^\mu,\frac{\partial x^m}{\partial z^\nu}\right]_-=2i\sigma_{ab}L^2\frac{\partial e^m_a}{\partial x^\nu}\frac{\partial e^\mu_b}{\partial x^\rho}dx^\rho$$

Substituting this into (53), we have

$$\begin{aligned}d\tau_q^2&=g_{nm}(x)\frac{\partial x^n}{\partial z^\mu}\frac{\partial x^m}{\partial z^\nu}dz^\mu dz^\nu\\&\quad+g_{nm}(x)\frac{\partial x^n}{\partial z^\mu}2i\sigma_{ab}L^2\frac{\partial e^m_a}{\partial x^\nu}\frac{\partial e^\mu_b}{\partial x^\rho}dx^\rho dz^\nu\end{aligned}$$

Due to (42), the last term goes to zero with an accuracy of L^3. Thus,

$$d\tau_q^2 = g_{\mu\nu}(z)\, dz^\mu\, dz^\nu \tag{54}$$

where $g_{\mu\nu}(z)$ is the metric tensor in quantum space-time, which is defined as

$$g_{\mu\nu}(z) = g_{nm}(x) \frac{\partial x^n}{\partial z^\mu} \frac{\partial x^m}{\partial z^\nu} \tag{55}$$

For the photon or neutrino, the proper time (54) acquires the form

$$g_{\mu\nu}(z) \frac{dz^\mu}{d\sigma_q} \frac{dz^\nu}{d\sigma_q} = 0 \tag{56}$$

where $\sigma_q = z^0$.

Finally, we express $d\tau_q^2$ through the usual proper time $d\tau^2 = g_{nm}(x)\, dx^n\, dx^m$ in an arbitrary gravitational field. One can easily obtain its explicit form if representations (1) and (42) are used. Thus, in the tetrad formalism it reads

$$d\tau_q^2 = g_{nm}(x)\, dx^n\, dx^m + 4iL^2 \sigma_{ab} g_{nm}(x) \frac{\partial e_a^\rho}{\partial x^\delta} \frac{\partial e_b^n}{\partial x^\rho}\, dx^\delta\, dx^m \tag{57}$$

This expression will be used below.

3.4. Quantization of the Metric Tensor

First, we notice that $g_{\mu\nu}(z)$ is the covariant tensor in quantum space-time. Indeed, let

$$g_{\mu\nu}(x) = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}$$

where ξ^α are local inertial coordinates. Then, in any system of reference z^μ the metric tensor is given by

$$g'_{\mu\nu}(z) = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial z^\mu} \frac{\partial \xi^\beta}{\partial z^\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial x^\rho}{\partial z^\mu} \frac{\partial \xi^\beta}{\partial x^\delta} \frac{\partial x^\delta}{\partial z^\nu}$$

and therefore

$$g'_{\mu\nu} = g_{\rho\delta}(x) \frac{\partial x^\rho}{\partial z^\mu} \frac{\partial x^\delta}{\partial z^\nu}$$

From this, we see that $g_{\mu\nu}(z)$ is indeed the covariant tensor. In the quantum space-time case, the choice of its inverse tensor is somewhat difficult. If, by analogy with the usual theory, we use the definition

$$g^{\lambda\mu}(z) \equiv \frac{\partial z^\lambda}{\partial \xi^\alpha}\frac{\partial z^\mu}{\partial \xi^\beta}\eta^{\alpha\beta} = \frac{\partial z^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial \xi^\alpha}\frac{\partial z^\mu}{\partial x^\delta}\frac{\partial x^\delta}{\partial \xi^\beta}\eta^{\alpha\beta}$$
$$= g^{\rho\delta}(x)\frac{\partial z^\lambda}{\partial x^\rho}\frac{\partial z^\mu}{\partial x^\delta} \tag{58}$$

then

$$g^{\lambda\mu}(z)g_{\mu\nu}(z) = \frac{\partial z^\lambda}{\partial x^\rho}\frac{\partial z^\mu}{\partial x^\sigma}g^{\rho\sigma}(x)\frac{\partial x^\delta}{\partial z^\mu}\frac{\partial x^\eta}{\partial z^\nu}g_{\delta\eta}(x)$$

Making use of the last equality in definition (43), we get

$$g^{\lambda\mu}(z)g_{\mu\nu}(z) = \frac{\partial z^\lambda}{\partial x^\rho}\frac{\partial x^\eta}{\partial z^\nu}g^{\rho\delta}(x)g_{\delta\eta}(x)$$
$$= \frac{\partial z^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial z^\nu} = \delta^\lambda_\nu + 2iL^2\sigma_{ab}\frac{\partial e^\rho_a}{\partial x^\nu}\frac{\partial e^\lambda_b}{\partial x^\rho}$$

Here we have again used definition (43) and the standard relation

$$g^{\rho\delta}(x)g_{\delta\eta}(x) = \delta^\rho_\eta$$

Analogously, taking into account definition (58), one can obtain

$$g_{\lambda\mu}(z)g^{\mu\nu}(z) = \delta^\nu_\lambda + 2iL^2\sigma_{ab}\frac{\partial e^q_a}{\partial x^\rho}\frac{\partial e^\eta_b}{\partial x^q}g_{\lambda\eta}(x)g^{\rho\nu}(x)$$
$$= \delta^\nu_\lambda + 2iL^2\sigma_{ab}\frac{\partial e^q_a}{\partial x^\nu}\frac{\partial e^\lambda_b}{\partial x^q}$$

Collecting these results together, we have

$$g^{\lambda\mu}(z)g_{\mu\nu}(z) - g_{\lambda\mu}(z)g^{\mu\nu}(z) = \delta^\lambda_\nu - \delta^\nu_\lambda \tag{59}$$

Thus, definition (58) does not satisfy the condition of invertibility of $g^{\lambda\mu}(z)$. If instead of (58) we define another representation

$$g^{\sigma\nu}(z) = g^{\delta n}(x)\frac{\partial z^\sigma}{\partial x^\delta}\frac{\partial z^\nu}{\partial x^n} + L^2 g^{\nu m}\left(\frac{\partial \Pi^\sigma}{\partial x^q}\frac{\partial \Pi^q}{\partial x^m} - \frac{\partial \Pi^q}{\partial x^m}\frac{\partial \Pi^\sigma}{\partial x^q}\right) \tag{60}$$

then it is easily verified that

$$g^{\sigma\nu}(z)g_{\nu\rho}(z) = \delta^\sigma_\rho, \qquad g_{\sigma\nu}(z)g^{\nu\rho}(z) = \delta^\rho_\sigma \tag{61}$$

However, because of the noncommutability of $g_{\mu\nu}(z)$ with respect to indices μ and ν, for any case it is optional that relation (61) obtained by means of (60) preserves its form, for example,

$$g^{\sigma\nu}(z)g_{\rho\nu}(z)=\delta^{\sigma}_{\rho}+L^2g^{\sigma\nu}(x)g_{\kappa\eta}(x)\left(\frac{\partial\Pi^{\varkappa}}{\partial x^{\rho}}\frac{\partial\Pi^{\eta}}{\partial x^{\nu}}-\frac{\partial\Pi^{\eta}}{\partial x^{\nu}}\frac{\partial\Pi^{\varkappa}}{\partial x^{\rho}}\right)$$

$$g^{\nu\sigma}(z)g_{\nu\rho}(z)=\delta^{\sigma}_{\rho}+L^2\left(\frac{\partial\Pi^{q}}{\partial x^{\rho}}\frac{\partial\Pi^{\sigma}}{\partial x^{q}}-\frac{\partial\Pi^{\sigma}}{\partial x^{q}}\frac{\partial\Pi^{q}}{\partial x^{\rho}}\right)+L^2g^{\delta n}(x)g_{\nu\rho}(x)$$
$$\times\left(\frac{\partial\Pi^{\nu}}{\partial x^{\delta}}\frac{\partial\Pi^{\sigma}}{\partial x^{n}}-\frac{\partial\Pi^{\sigma}}{\partial x^{n}}\frac{\partial\Pi^{\nu}}{\partial x^{\delta}}\right)$$
$$+L^2g^{m\sigma}(x)g_{\nu\rho}(x)\left(\frac{\partial\Pi^{\nu}}{\partial x^{q}}\frac{\partial\Pi^{q}}{\partial x^{m}}-\frac{\partial\Pi^{q}}{\partial x^{m}}\frac{\partial\Pi^{\nu}}{\partial x^{q}}\right)$$

$$g_{\nu\sigma}(z)g^{\nu\rho}(z)=\delta^{\rho}_{\sigma}+L^2g_{\eta\varkappa}(x)g^{\nu\rho}(x)\left(\frac{\partial\Pi^{\varkappa}}{\partial x^{\nu}}\frac{\partial\Pi^{\eta}}{\partial x^{\sigma}}-\frac{\partial\Pi^{\eta}}{\partial x^{\sigma}}\frac{\partial\Pi^{\varkappa}}{\partial x^{\nu}}\right)$$

etc.

It is natural that in our model the metric tensor consists of two parts; a symmetric part

$$g^{s}_{\mu\nu}(z)=\frac{1}{2}g_{nm}(x)\left(\frac{\partial x^n}{\partial z^{\mu}}\frac{\partial x^m}{\partial z^{\nu}}+\frac{\partial x^n}{\partial z^{\nu}}\frac{\partial x^m}{\partial z^{\mu}}\right) \tag{62}$$

and an antisymmetric part

$$g^{a}_{\mu\nu}(z)=\frac{1}{2}g_{nm}(x)\left(\frac{\partial x^n}{\partial z^{\mu}}\frac{\partial x^m}{\partial z^{\nu}}-\frac{\partial x^n}{\partial z^{\nu}}\frac{\partial x^m}{\partial z^{\mu}}\right)$$
$$=iL^2\sigma_{ab}g_{nm}(x)\frac{\partial e^n_a}{\partial x^{\mu}}\frac{\partial e^m_b}{\partial x^{\nu}} \tag{63}$$

Now we find the commutator of $g_{\mu\nu}(z)$ at different points x^{μ} and y^{μ} of space-time on the large scale. Thus,

$$z^{\mu}_x=x^{\mu}+L\Pi^{\mu}(x),\qquad z^{\mu}_y=y^{\mu}+L\Pi^{\mu}(y)$$

It is expected that according to translation invariance the commutator

$$[g_{\mu\nu}(z_x),g_{\mu\nu}(z_y)]_-=F(x-y) \tag{64}$$

should be a function of the variables $x^{\mu}-y^{\mu}$. From a physical point of view, after some averaging procedure (this is similar to taking expectation values over the vacuum state in quantum field theory) the averaged function $\langle\cdots|F(x-y)|\cdots\rangle$ can be interpreted as a wave function or particle propagator transmitted gravitational signal between points x^{μ} and y^{μ}. We show here that even for the simple case (5) this assertion holds.

Thus, inserting the inverse operation (42) in the explicit form (55) of $g_{\mu\nu}(z)$, we have

$$g_{\mu\nu}(z_x)=g_{\mu\nu}(x)-L\frac{\partial\Pi^\nu}{\partial x^\mu}-L\frac{\partial\Pi^\mu}{\partial x^\nu}+L^2\frac{\partial\Pi^q}{\partial x^\mu}\frac{\partial\Pi^\nu}{\partial x^q}$$
$$+L^2\frac{\partial\Pi^m}{\partial x^\mu}\frac{\partial\Pi^m}{\partial x^\nu}+L^2\frac{\partial\Pi^q}{\partial x^\nu}\frac{\partial\Pi^\mu}{\partial x^q}$$

Multiplying this expression by $g_{\mu\nu}(z_y)$ from the right-hand side and taking the difference between this and the inverted multiplier, one obtains

$$[g_{\mu\nu}(z_x)g_{\mu\nu}(z_y)-g_{\mu\nu}(z_y)g_{\mu\nu}(z_x)]=2L^2I$$

where

$$I=2i\sigma_{ab}\left[\frac{\partial e_a^\nu(x)}{\partial x^\mu}\frac{\partial e_b^\nu(y)}{\partial y^\mu}+\frac{\partial e_a^\mu(x)}{\partial x^\nu}\frac{\partial e_b^\nu(y)}{\partial y^\mu}\right]$$

Consider two-dimensional space-time $\nu,\mu=0,1$ and use the tetrad field (5). Then, after simple calculation we have

$$[g_{\mu\nu}(z_x),g_{\mu\nu}(z_y)]_-=2L^2I \tag{65}$$
$$I=6i\sigma_{01}(\omega^2/c^2+k^2)\sin[\omega(t_x-t_y)-k(x-y)]$$
$$\sigma_{01}=(1/2i)(\gamma_0\gamma_1-\gamma_1\gamma_0)$$

From the definition (65) one can choose the vacuum-like state in quantum space–time, and the peculiarity of its structural property is that it has spinor character satisfying the following conditions:

$$\bar{u}_0u_0=1,\qquad \bar{u}_0\gamma_0\gamma_1u_0=1 \tag{66}$$

For example, assuming

$$u_0=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\qquad \bar{u}_0=u^*\gamma_0=(1,0,0,0)$$

we get

$$\langle u_0|[g_{\mu\nu}(z_x),g_{\mu\nu}(z_y)]_-|u_0\rangle$$
$$=12(\omega^2/c^2+k^2)\sin[\omega(t_x-t_y)-k(x-y)] \tag{67}$$

We notice that in the more general case $[g_{\mu\nu}(z_x),g_{\lambda\rho}(z_y)]$ should be proportional to L^2 and gives a fourth-rank tensor $G_{\mu\nu,\lambda\rho}(x-y)$. It seems that a tensor of this type, after lowering some indices $\mu\nu$, $\lambda\rho$, determines the propagator of the graviton-like field.

3.5. Connection between $g_{\mu\nu}(z)$ and $\Gamma^{\lambda}_{\mu\nu}(z)$

As shown above, in quantum space-time, the field defining gravitational force is expressed through the "affine connection" $\Gamma^{\lambda}_{\mu\nu}(z)$, whereas the proper time interval is given by the "metric tensor" $g_{\mu\nu}(z)$. Now we show that $g_{\mu\nu}(z)$ is also the gravitational potential, i.e., its derivative gives the field $\Gamma^{\lambda}_{\mu\nu}(z)$ up to $O(L^2)$.

We recall that the metric tensor is given by expression (55),

$$g_{\mu\nu}(z) = g_{nm}(x)\frac{\partial x^n}{\partial z^\mu}\frac{\partial x^m}{\partial z^\nu}$$

or

$$g_{\mu\nu}(z) = \eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial z^\mu}\frac{\partial \xi^\beta}{\partial z^\nu}$$

Differentiation of the last term with respect to z^λ yields

$$I_1 = \frac{\partial g_{\mu\nu}(z)}{\partial z^\lambda} = \eta_{\alpha\beta}\frac{\partial x^\rho}{\partial z^\lambda}\frac{\partial}{\partial x^\rho}\left(\frac{\partial \xi^\alpha}{\partial z^\mu}\frac{\partial \xi^\beta}{\partial z^\nu}\right)$$
$$= \eta_{\alpha\beta}\frac{\partial x^\rho}{\partial z^\lambda}\left[\frac{\partial}{\partial x^\rho}\left(\frac{\partial \xi^\alpha}{\partial z^\mu}\right)\frac{\partial \xi^\beta}{\partial z^\nu} + \frac{\partial \xi^\alpha}{\partial z^\mu}\frac{\partial}{\partial x^\rho}\left(\frac{\partial \xi^\beta}{\partial z^\nu}\right)\right]$$

Further, making use of the commutator property

$$\left[\frac{\partial x^\rho}{\partial z^\lambda}, \frac{\partial \xi^\alpha}{\partial z^\mu}\right] = L^2\frac{\partial \xi^\alpha}{\partial x^\delta} I^{\rho\delta}_{\lambda\mu}$$

we obtain

$$I_1 = \eta_{\alpha\beta}\left(\frac{\partial^2 \xi^\alpha}{\partial z^\lambda\,\partial z^\mu}\frac{\partial \xi^\beta}{\partial z^\nu} + \frac{\partial \xi^\alpha}{\partial z^\mu}\frac{\partial^2 \xi^\beta}{\partial z^\lambda\,\partial z^\nu} + L^2 I^{\rho\delta}_{\lambda\mu}\frac{\partial \xi^\alpha}{\partial x^\delta}\frac{\partial^2 \xi^\beta}{\partial x^\nu\,\partial x^\rho}\right) \tag{68}$$

where

$$I^{\rho\delta}_{\lambda\mu} = \frac{\partial \Pi^\rho}{\partial x^\lambda}\frac{\partial \Pi^\delta}{\partial x^\mu} - \frac{\partial \Pi^\delta}{\partial x^\mu}\frac{\partial \Pi^\rho}{\partial x^\lambda} \tag{69}$$

Now taking into account the definition (47) for $\Gamma^{\lambda}_{\mu\nu}(z)$ and equalities (43), one obtains the following relation:

$$\frac{\partial \xi^\beta}{\partial z^\lambda}\Gamma^{\lambda}_{\mu\nu}(z) = \frac{\partial^2 \xi^\beta}{\partial z^\mu\,\partial z^\nu} + L^2 A^\beta_\alpha\frac{\partial^2 \xi^\alpha}{\partial x^\mu\,\partial x^\nu} \tag{70}$$

where

$$A^\beta_\alpha = a^\rho_q\frac{\partial \xi^\beta}{\partial x^\rho}\frac{\partial x^q}{\partial \xi^\alpha}, \qquad a^\rho_q = 2i\sigma_{ab}\frac{\partial e^n_a}{\partial x^q}\frac{\partial e^\rho_b}{\partial x^n}$$

In the usual theory of gravity one has the well-known relation

$$\frac{\partial^2 \xi^\rho}{\partial x^\lambda\, \partial x^\mu} = \frac{\partial \xi^\rho}{\partial x^q} \left\{ \begin{matrix} q \\ \lambda\mu \end{matrix} \right\}$$

where $\{{}^{q}_{\lambda\mu}\}$ is the usual affine connection given above. Then, expression (70) can be rewritten in the form

$$\frac{\partial^2 \xi^\beta}{\partial z^\mu\, \partial z^\nu} = \frac{\partial \xi^\beta}{\partial z^\rho} \Gamma^\rho_{\mu\nu}(z) - L^2 I^\rho_q \frac{\partial \xi^\beta}{\partial x^\rho} \left\{ \begin{matrix} q \\ \mu\nu \end{matrix} \right\} \tag{71}$$

where

$$I^\rho_q = \frac{\partial \Pi^n}{\partial x^q} \frac{\partial \Pi^\rho}{\partial x^n} - \frac{\partial \Pi^\rho}{\partial x^n} \frac{\partial \Pi^n}{\partial x^q} \tag{72}$$

Substituting equality (70) into (68), we have

$$\frac{\partial g_{\mu\nu}(z)}{\partial z^\lambda} = \frac{\partial \xi^\alpha}{\partial z^\rho} \Gamma^\rho_{\lambda\mu} \frac{\partial \xi^\beta}{\partial z^\nu} \eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial z^\mu} \frac{\partial \xi^\beta}{\partial z^\rho} \Gamma^\rho_{\lambda\nu} \eta_{\alpha\beta} + L^2 N^\lambda_{\mu\nu} \tag{73}$$

where

$$N^\lambda_{\mu\nu} = I^{\rho\delta}_{\lambda\mu} g_{\delta q}(x) \left\{ \begin{matrix} q \\ \nu\rho \end{matrix} \right\} - I^\rho_q g_{\nu\rho}(x) \left\{ \begin{matrix} q \\ \lambda\mu \end{matrix} \right\} - I^\rho_q g_{\mu\rho}(x) \left\{ \begin{matrix} q \\ \lambda\nu \end{matrix} \right\}$$

Now one should define the commutator $[\Gamma^\rho_{\lambda\mu}, \partial\xi^\beta/\partial z^\nu]_-$. By definition,

$$\frac{\partial \xi^\beta}{\partial z^\nu} \Gamma^\rho_{\lambda\mu} = \frac{\partial \xi^\beta}{\partial z^\nu} \frac{\partial z^\rho}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial z^\lambda\, \partial z^\mu} = \frac{\partial z^\rho}{\partial \xi^\alpha} a^{\beta\alpha}_{\nu\lambda\mu} - L^2 I^{i\rho}_{\nu q} \frac{\partial \xi^\beta}{\partial x^i} \left\{ \begin{matrix} q \\ \lambda\mu \end{matrix} \right\} \tag{74}$$

where we have used the commutator

$$\left[\frac{\partial x^i}{\partial z^\nu}, \frac{\partial z^\rho}{\partial x^j} \right]_- = -L^2 I^{i\rho}_{\nu j}$$

and

$$a^{\beta\alpha}_{\nu\lambda\mu} = \frac{\partial \xi^\beta}{\partial z^\nu} \frac{\partial^2 \xi^\alpha}{\partial z^\lambda\, \partial z^\mu} = \frac{\partial \xi^\beta}{\partial z^\nu} \frac{\partial x^\rho}{\partial z^\lambda} \frac{\partial}{\partial x^\rho} \left(\frac{\partial \xi^\alpha}{\partial z^\mu} \right) \tag{75}$$

We note that our calculation procedure in (74) and (75) has consisted in moving $\partial\xi^\beta/\partial z^\nu$ through $\Gamma^\rho_{\lambda\mu}(z)$. Thus, in (75) we again use the commutator

$$\frac{\partial \xi^\beta}{\partial x^\rho} \left[\frac{\partial x^\rho}{\partial z^\nu}, \frac{\partial x^\varkappa}{\partial z^\lambda} \right]_- = L^2 \frac{\partial \xi^\beta}{\partial x^\rho} I^{\rho\varkappa}_{\nu\lambda}$$

and the result reads

$$a^{\beta\alpha}_{\nu\lambda\mu} = \frac{\partial x^\varkappa}{\partial z^\lambda} b^{\beta\alpha}_{\nu\varkappa\mu} + L^2 I^{i\varkappa}_{\nu\lambda} \frac{\partial \xi^\beta}{\partial x^i} \frac{\partial^2 \xi^\alpha}{\partial x^\varkappa\, \partial x^\mu}$$

Here the term

$$b^{\beta\alpha}_{\nu\varkappa\mu} = \frac{\partial\xi^\beta}{\partial z^\nu}\frac{\partial}{\partial x^\varkappa}\left(\frac{\partial\xi^\alpha}{\partial z^\mu}\right)$$

is in turn reduced to

$$b^{\beta\alpha}_{\nu\varkappa\mu} = c^{\beta\alpha}_{\nu\varkappa\mu} + \frac{\partial\xi^\beta}{\partial z^\nu}\frac{\partial x^m}{\partial z^\mu}\frac{\partial^2\xi^\alpha}{\partial x^\varkappa\,\partial x^m}$$

After some calculation the first term of this expression acquires the form

$$\begin{aligned} c^{\beta\alpha}_{\nu\varkappa\mu} &\equiv \frac{\partial\xi^\beta}{\partial z^\nu}\frac{\partial}{\partial x^\varkappa}\left(\frac{\partial x^m}{\partial z^\mu}\right)\frac{\partial\xi^\alpha}{\partial x^m} \\ &= \left[\frac{\partial}{\partial x^\varkappa}\left(\frac{\partial x^m}{\partial z^\mu}\right)\frac{\partial\xi^\beta}{\partial z^\nu} + \frac{\partial x^m}{\partial z^\mu}\frac{\partial^2\xi^\beta}{\partial x^\varkappa\,\partial z^\nu}\right. \\ &\quad \left. + L^2\frac{\partial}{\partial x^\varkappa}\left(\frac{\partial\xi^\beta}{\partial x^k}I^{km}_{\nu\mu}\right) - \frac{\partial^2\xi^\beta}{\partial x^\varkappa\,\partial z^\nu}\frac{\partial x^m}{\partial z^\mu}\right]\frac{\partial\xi^\alpha}{\partial x^m} \end{aligned}$$

Collecting these cycle equalities together, we have

$$\begin{aligned} &\left[\frac{\partial\xi^\beta}{\partial z^\nu}, \Gamma^\rho_{\lambda\mu}(z)\right]_- \\ &= L^2\left(-I^{i\rho}_{\nu q}\frac{\partial\xi^\beta}{\partial x^i}\begin{Bmatrix} q \\ \lambda\mu \end{Bmatrix} + I^{i\varkappa}_{\nu\lambda}\frac{\partial\xi^\beta}{\partial x^i}\begin{Bmatrix} \rho \\ \varkappa\mu \end{Bmatrix}\right. \\ &\quad \left. + I^{\rho m}_{\mu\nu}\frac{\partial\xi^\beta}{\partial x^q}\begin{Bmatrix} q \\ \lambda m \end{Bmatrix} + I^{km}_{\nu\mu}\frac{\partial\xi^\beta}{\partial x^k}\begin{Bmatrix} \rho \\ \lambda m \end{Bmatrix} + \frac{\partial}{\partial x^\lambda}Q^\beta_{\rho,\nu\mu}\right) \end{aligned} \tag{76}$$

where

$$Q^\beta_{\rho,\nu\mu} = \frac{\partial\xi^\beta}{\partial x^k}I^{k\rho}_{\nu\mu}$$

Finally, due to expression (76), equality (73) takes the form

$$\frac{\partial g_{\mu\nu}(z)}{\partial z^\lambda} = g_{\rho\nu}(z)\Gamma^\rho_{\lambda\mu}(z) + g_{\mu\rho}\Gamma^\rho_{\lambda\nu}(z) + L^2 D^\lambda_{\mu\nu} \tag{77}$$

where

$$\begin{aligned} D^\lambda_{\mu\nu} &= I^{\rho\delta}_{\lambda\mu}g_{\delta q}(x)\begin{Bmatrix} q \\ \nu\rho \end{Bmatrix} - I^\rho_q g_{\nu\rho}(x)\begin{Bmatrix} q \\ \lambda\mu \end{Bmatrix} - I^\rho_q g_{\mu\rho}\begin{Bmatrix} q \\ \lambda\nu \end{Bmatrix} \\ &\quad + I^{i\rho}_{\nu q}g_{i\rho}(x)\begin{Bmatrix} q \\ \lambda\mu \end{Bmatrix} - I^{i\varkappa}_{\nu\lambda}g_{\rho i}(x)\begin{Bmatrix} \rho \\ \varkappa\mu \end{Bmatrix} - (I^{km}_{\mu\nu} + I^{km}_{\nu\mu}) \\ &\quad \times g_{kq}(x)\begin{Bmatrix} q \\ \lambda m \end{Bmatrix} - I^{km}_{\nu\mu}g_{mq}(x)\begin{Bmatrix} q \\ \lambda k \end{Bmatrix} - g_{km}(x)\frac{\partial}{\partial x^\lambda}I^{km}_{\nu\mu} \end{aligned}$$

In order to express the affine connection $\Gamma^\lambda_{\mu\nu}(z)$ through the metric tensor $g_{\mu\nu}(z)$, we add to (77) the analogous relation with rearranged indices μ and λ and subtract from (77) the analogous relation with rearranged indices ν and λ. As a result, we get

$$\begin{aligned}\frac{\partial g_{\mu\nu}(z)}{\partial z^\lambda}+\frac{\partial g_{\lambda\nu}(z)}{\partial z^\mu}-\frac{\partial g_{\mu\lambda}(z)}{\partial z^\nu}
&= g_{\rho\nu}(z)\Gamma^\rho_{\lambda\mu}(z)+g_{\mu\rho}(z)\Gamma^\rho_{\lambda\nu}(z)\\
&\quad+g_{\rho\nu}(z)\Gamma^\rho_{\mu\lambda}(z)+g_{\lambda\rho}(z)\Gamma^\rho_{\mu\nu}(z)-g_{\rho\lambda}(z)\Gamma^\rho_{\nu\mu}(z)-g_{\mu\rho}(z)\Gamma^\rho_{\nu\lambda}(z)\\
&\quad+L^2(D^\lambda_{\mu\nu}+D^\mu_{\lambda\nu}-D^\nu_{\mu\lambda})\end{aligned}\tag{78}$$

Further, for terms of the type $g_{\rho\nu}(z)\Gamma^\rho_{\lambda\mu}(z)$ on the right-hand side of (78) their symmetric and antisymmetric parts should be separated by using relations (48), (62), and (63); for example,

$$g_{\rho\nu}(z)\Gamma^\rho_{\lambda\mu}(z)=(g^s_{\rho\nu}+L^2g^a_{\rho\nu})(\Gamma^\rho_{s,\lambda\mu}+L^2\Gamma^\rho_{a,\lambda\mu}),$$

Thus, all symmetric parts give $2g^s_{\rho\nu}(z)\Gamma^\rho_{s,\lambda\mu}(z)$ and the remaining mixed terms $g^a_{\rho\nu}(z)\Gamma^\rho_{s,\lambda\mu}(z)+g^s_{\rho\nu}(z)\Gamma^\rho_{a,\lambda\mu}(z)$, etc., should be joined with terms $D^\mu_{\lambda\nu},\ldots$. As a result, we have

$$\frac{\partial g_{\mu\nu}(z)}{\partial z^\lambda}+\frac{\partial g_{\lambda\nu}(z)}{\partial z^\mu}-\frac{\partial g_{\mu\lambda}(z)}{\partial z^\nu}=2g^s_{\rho\nu}(z)\Gamma^\rho_{s,\lambda\mu}(z)+L^2\delta^\lambda_{\mu\nu}\tag{79}$$

where $\delta^\lambda_{\mu\nu}$ is an expression of the type of $D^\lambda_{\mu\nu}$ and its explicit form will be presented below.

Now, on the right-hand side of (79) we return again to the complete expression for $g_{\rho\nu}(z)$ and $\Gamma^\rho_{\lambda\mu}(z)$,

$$\begin{aligned}2g^s_{\rho\nu}(z)\Gamma^\rho_{s,\lambda\mu}(z)+L^2\delta^\lambda_{\mu\nu}
&=2[g_{\rho\nu}(z)-L^2g^a_{\rho\nu}(z)][\Gamma^\rho_{\lambda\mu}(z)-L^2\Gamma^\rho_{a,\lambda\mu}]+L^2\delta^\lambda_{\mu\nu}\\
&=2g_{\rho\nu}(z)\Gamma^\rho_{\lambda\mu}(z)+L^2\left[\delta^\lambda_{\mu\nu}-g_{nm}(x)I^{nm}_{\rho\nu}\left\{\begin{matrix}\rho\\ \lambda\mu\end{matrix}\right\}-g_{\rho\nu}(x)I^{\delta k}_{\lambda\mu}\left\{\begin{matrix}\rho\\ \delta k\end{matrix}\right\}\right]\end{aligned}\tag{80}$$

where

$$g^a_{\rho\nu}(z)=\frac{1}{2}g_{nm}I^{nm}_{\rho\nu},\qquad \Gamma^\rho_{a,\lambda\mu}(z)=\frac{1}{2}I^{\delta k}_{\lambda\mu}\left\{\begin{matrix}\rho\\ \delta k\end{matrix}\right\}$$

Substituting (80) into (79) and multiplying the obtained expression by $g^{\sigma\nu}(z)$ from the left, we get

$$\Gamma^\sigma_{\lambda\mu}(z)=\frac{1}{2}g^{\sigma\nu}(z)\left(\frac{\partial g_{\mu\nu}(z)}{\partial z^\lambda}+\frac{\partial g_{\lambda\nu}(z)}{\partial z^\mu}-\frac{\partial g_{\mu\lambda}(z)}{\partial z^\nu}\right)-\frac{L^2}{2}\Delta^\sigma_{\lambda\mu}\tag{81}$$

where

$$\Delta^{\sigma}_{\lambda\mu}=2g^{\sigma\nu}(x)g_{nm}(x)I^{nm}_{\rho\nu}\left\{ {\rho \atop \lambda\mu} \right\}-g^{\sigma\nu}(x)g_{nm}(x)I^{nm}_{\rho\nu}\left\{ {\rho \atop \lambda\mu} \right\}$$

$$-g^{\sigma\nu}(x)g_{\rho\nu}(x)I^{\delta k}_{\lambda\mu}\left\{ {\rho \atop \delta k} \right\}+g^{\sigma\nu}(x)\delta^{\lambda}_{\mu\nu}$$

$$=g^{\sigma\nu}(x)M^{\lambda}_{\mu\nu}$$

$$M^{\lambda}_{\mu\nu}=I^{\delta k}_{\lambda\nu}g_{\mu\rho}(x)\left\{ {\rho \atop \delta k} \right\}+I^{\delta k}_{\mu\nu}g_{\lambda\rho}(x)\left\{ {\rho \atop \delta k} \right\}$$

$$-I^{\delta k}_{\lambda\mu}g_{\rho\nu}(x)\left\{ {\rho \atop \delta k} \right\}+I^{\rho\delta}_{\lambda\mu}g_{\delta q}(x)\left\{ {q \atop \nu\rho} \right\}$$

$$-2g_{\nu m}(x)I^{m}_{\rho}\left\{ {\rho \atop \lambda\mu} \right\}+g_{\rho i}(x)\left\{ {\rho \atop \varkappa\mu} \right\}(I^{i\varkappa}_{\lambda\nu}-2I^{i\varkappa}_{\nu\lambda})$$

$$+g_{\rho i}(x)\left\{ {\rho \atop \varkappa\lambda} \right\}(I^{i\varkappa}_{\mu\nu}-2I^{i\varkappa}_{\nu\mu})+2g_{km}(x)\left(\frac{\partial\Pi^k}{\partial x^\lambda}\frac{\partial^2\Pi^m}{\partial x^\nu\,\partial x^\mu}\right.$$

$$\left.-\frac{\partial^2\Pi^m}{\partial x^\nu\,\partial x^\mu}\frac{\partial\Pi^k}{\partial x^\lambda}+\frac{\partial^2\Pi^m}{\partial x^\mu\,\partial x^\lambda}\frac{\partial\Pi^k}{\partial x^\nu}-\frac{\partial\Pi^k}{\partial x^\nu}\frac{\partial^2\Pi^m}{\partial x^\mu\,\partial x^\lambda}\right)$$

Here we use the definition

$$g^{\sigma\nu}(z)g_{\rho\nu}(z)=\delta^{\sigma}_{\rho}+L^2g^{\sigma\nu}(x)g_{\varkappa\eta}(x)I^{\varkappa\eta}_{\rho\nu}$$

Thus, we see that to the accuracy of the L^2 term, $g_{\mu\nu}(z)$ is also the grivatational potential in quantum space-time, i.e., its derivative defines the field $\Gamma^{\lambda}_{\mu\nu}(z)$ by formula (81).

Finally, we note that the antisymmetric property of the metric tensor $g_{\mu\nu}(z)$ and also the affine connection $\Gamma^{\lambda}_{\mu\nu}(z)$ are connected with the existence of the torsion tensor. It seems to us that in the whole space-time the latter gives rise to a slight violation of the homogeneity and isotropy of the universe, or in the language of particle motion, in the inertial system of reference the law of motion acquires some change in accordance with (27).

3.6. The Newtonian Approximation

Consider the motion equation (46) and find its Newtonian version. Let a particle move slowly in a weak stationary gravitational field. Before investigating this equation it is necessary to discuss some problems concerning proper time defined by formula (57). First, for calculational convenience in equation (46) differentiation with respect to $d\tau_q$ should be replaced by $d\tau=[g_{nm}(x)\,dx^n\,dx^m]^{1/2}$ (here we use a system of units in which $c=1$ and

$x_0 = t$). This is achieved by the following substitutions:

$$\begin{aligned}\frac{d}{d\tau_q} &= \frac{d}{d\tau}\left(\frac{d\tau_q}{d\tau}\right)^{-1}\\ \frac{d^2}{d\tau_q^2} &= \frac{d^2}{d\tau^2}\left(\frac{d\tau_q}{d\tau}\right)^{-2} - \frac{d}{dt}\left(\frac{d\tau_q}{d\tau}\right)^{-2}\frac{d^2\tau_q}{d\tau^2}\left(\frac{d\tau_q}{d\tau}\right)^{-1}\end{aligned} \tag{82}$$

where

$$d\tau_q = \left[d\tau^2 + 4iL^2\sigma_{ab}g_{nm}(x)\frac{\partial e_a^\rho}{\partial x^\delta}\frac{\partial e_b^n}{\partial x^\rho}\,dx^\delta\,dx^m\right]^{1/2}$$

or

$$\frac{d\tau_q}{d\tau} = \left[1 + 4iL^2\sigma_{ab}g_{nm}(x)\frac{\partial e_a^\rho}{\partial x^\delta}\frac{\partial e_b^n}{\partial x^\rho}\frac{dx^\delta}{d\tau}\frac{dx^m}{d\tau}\right)\right]^{1/2}$$

If the particle is sufficiently slow, one can neglect $dx^i/d\tau$ $(i = 1, 2, 3)$ with respect to $dt/d\tau$ and write

$$\begin{aligned}(d\tau_q/d\tau)^{-1} &= [1 + I(dt/d\tau)^2]^{-1/2} = 1 - \tfrac{1}{2}I(dt/d\tau)^2\\ (d\tau_q/d\tau)^{-2} &= 1 - I(dt/d\tau)^2\\ \frac{d^2\tau_q}{d\tau^2} &= \frac{1}{2}\left[1 - \frac{1}{2}I\left(\frac{dt}{d\tau}\right)^2\right]\\ &\quad\times\left[4iL^2\sigma_{ab}\frac{\partial g_{n0}(x)}{\partial x^\lambda}\frac{dx^\lambda}{d\tau}\frac{\partial e_a^\rho}{\partial t}\frac{\partial e_b^n}{\partial x^\rho}\left(\frac{dt}{d\tau}\right)^2\right.\\ &\quad + 4iL^2\sigma_{ab}g_{n0}(x)\frac{\partial^2 e_a^\rho}{\partial t\,\partial x^\lambda}\frac{dx^\lambda}{d\tau}\frac{\partial e_b^n}{\partial x^\rho}\left(\frac{dt}{d\tau}\right)^2\\ &\quad\left. + 4iL^2\sigma_{ab}g_{n0}(x)\frac{\partial e_a^\rho}{\partial t}\frac{\partial^2 e_b^n}{\partial x^\rho\,\partial x^\lambda}\frac{dx^\lambda}{d\tau}\left(\frac{dt}{d\tau}\right)^2\right]\\ I &= 4i\sigma_{ab}L^2 g_{n0}(x)\frac{\partial e_a^\rho}{\partial t}\frac{\partial e_b^n}{\partial x^\rho}\end{aligned} \tag{83}$$

Thus, after the substitutions (82), equation (46) takes the form

$$\begin{aligned}&\frac{d^2z^\lambda}{d\tau^2}\left(\frac{d\tau_q}{d\tau}\right)^{-2} - \frac{dz^\lambda}{d\tau}\left(\frac{d\tau_q}{d\tau}\right)^{-2}\frac{d^2\tau_q}{d\tau^2}\left(\frac{d\tau_q}{d\tau}\right)^{-1}\\ &\quad + 2iL^2\sigma_{ab}\frac{\partial e_a^\rho}{\partial x^\mu}\frac{\partial e_b^\lambda}{\partial x^\rho}\frac{d^2z^\mu}{d\tau^2}\\ &\quad + \Gamma^\lambda_{\nu\mu}(z)\frac{dz^\nu}{d\tau}\left(\frac{d\tau_q}{d\tau}\right)^{-1}\frac{dz^\mu}{d\tau}\left(\frac{d\tau_q}{d\tau}\right)^{-1} = \frac{1}{m}f^\lambda(x)\end{aligned} \tag{84}$$

where the explicit form of $\Gamma^\lambda_{\nu\mu}(z)$ is given by expressions (48)-(50).

Now inserting these formulas for $\Gamma^{\lambda}_{\nu\mu}(z)$ into (84) and carrying out the averaging procedure (in the given case it is reduced to taking the trace of γ^a matrices), we obtain the following equation:

$$\frac{d^2x^\mu}{d\tau^2}+\begin{Bmatrix}\mu\\00\end{Bmatrix}\left(\frac{dt}{d\tau}\right)^2=\frac{1}{m}f^\mu_q(x) \tag{85}$$

where in the nonrelativistic case one can put $f^0(x)\equiv 0$. Further, we proceed according to the usual theory. Since the field is stationary, all time derivatives of $g_{\mu\nu}(x)$ disappear, and therefore

$$\begin{Bmatrix}\mu\\00\end{Bmatrix}=-\frac{1}{2}g^{\mu\nu}(x)\frac{\partial g_{00}}{\partial x^\nu}$$

Moreover, if the field is still weak, one can introduce an almost Cartesian system of coordinates in which

$$g_{\alpha\beta}=\eta_{\alpha\beta}+h_{\alpha\beta}, \qquad |h_{\alpha\beta}|\ll 1$$

Thus, to first order in $h_{\alpha\beta}$, one has

$$\begin{Bmatrix}\alpha\\00\end{Bmatrix}=-\frac{1}{2}\eta^{\alpha\beta}\frac{\partial h_{00}}{\partial x^\beta}$$

Substituting this expression for the usual affine connection into the motion equation (85), we get

$$\frac{d^2x^i}{d\tau^2}=\frac{1}{2}\left(\frac{dt}{d\tau}\right)^2\nabla^i h_{00}+\frac{1}{m}f^i_q, \qquad \frac{d^2t}{d\tau^2}=0 \tag{86}$$

where f^i_q is the nonrelativistic quantum force given by (13b) and (15b) for the general case and by (14a) and (14b) for the concrete cases. The solution of the second equation in (86) is $dt/d\tau=\text{const}$, and therefore

$$\frac{d^2x^i}{dt^2}=\frac{1}{2}\nabla^i h_{00}+\frac{1}{m}f^i_q \tag{87}$$

The corresponding Newtonian theory in accordance with equation (27) for quantum space-time gives

$$\frac{d^2x^i}{dt^2}=-\nabla^i\phi+\frac{1}{m}f^i_q \tag{88}$$

where ϕ is the gravitational potential, which for a spherical body with mass M is given by the formula

$$\phi=-GM/r$$

Comparing (87) with (88), one can conclude that

$$h_{00} = -2\phi + \text{const}$$

This is the usual result. So, assuming const = 0, we get

$$h_{00} = -2\phi, \qquad g_{00} = -(1+2\phi) \tag{89}$$

The gravitational potential is of the order of 10^{-39} on the "surface" of the proton, 10^{-9} on the surface of the Earth, 10^{-6} for the Sun, and 10^{-4} for white dwarf type stars.

4. CONSEQUENCES OF QUANTUM THEORY OF GRAVITY

4.1. Antigravitational Effect—Fifth Force in Quantum Space–Time

Now consider the very interesting possibility that the additional force caused by the quantum space–time structure gives rise to an antigravitational effect at least in the motion of quasiclassical particles for which energy and momentum satisfy the quantum mechanical relations $E = \hbar\omega$ and $p = \hbar k$. As shown above, for the concrete case (5), depending on the tetrad field property, the additional force (14a) or (14b) has two different natures: oscillatory and friction. We are interested in the friction case only, since the oscillating case requires numerical investigation and solution, which is beyond the scope of this paper. Our aim is to restudy the simple problem of the motion of a "freely" falling quasiclassical particle in the presence of the internal friction force in quantum space–time. In two-dimensional space–time on a large scale continued from the quantum one, the motion equation (88) for the problem of finding the height ($H = x$) of a falling particle in the Earth's acceleration field $g = 980 \text{ cm sec}^{-2}$ in the presence of the quantum friction force takes the form

$$\ddot{x} = -2L^2k^2\ddot{x} + g$$

where for a quasiclassical particle $k = mv/\hbar = m\dot{x}/\hbar$. Then this equation can be rewritten

$$\ddot{x} + \alpha\dot{x}^2\ddot{x} = g, \qquad \alpha = 2L^2m^2/\hbar^2 \tag{90}$$

To integrate this equation, we make a change of variable

$$\dot{x} = u(x), \qquad \ddot{x} = u'u, \qquad u' = du/dx$$

Thus, the equation for u acquires the form

$$u'(u + \alpha u^3) = g$$

or after simple integration

$$\tfrac{1}{2}u^2 + \tfrac{1}{4}\alpha u^4 = gx + c \tag{91}$$

Without loss of generality we assume $u(0)=0$, giving $c=0$. In our case $\alpha \ll 1$, and therefore we can easily obtain the solution of (91) and (90),

$$\frac{(2gx)^{1/2}}{g}\left(1+\frac{\alpha}{6}gx\right)=t+c_1 \tag{92}$$

where the integration constant is determined by the initial conditions

$$x(t)|_{t=0}=0 \qquad \text{and} \qquad x(t)|_{t=T}=H$$

In this case, the classical free fall time is

$$T=(2H/g)^{1/2}$$

However, due to equation (92) this time is changed and takes the value

$$t=T(1+\tfrac{1}{6}\alpha gH)$$

It is important to notice that by equation (90), the constant α depends on the mass of the particle. Therefore, the free fall time is not the same for particles of different weights. Let two particles with masses m_1 and m_2, say a proton m_p and an electron m_e, fall from height H; then their fall times are different:

$$t_p=T(1+\tfrac{1}{6}\alpha_p gH)$$
$$t_e=T(1+\tfrac{1}{6}\alpha_e gH)$$

where

$$\alpha_p=2L^2m_p^2/\hbar^2, \qquad \alpha_e=2L^2m_e^2/\hbar^2$$

Thus,

$$t_p-t_e=\Delta t=\tfrac{1}{6}gH(\alpha_p-\alpha_e)H$$

or

$$t_p/t_e=1+\tfrac{1}{6}gH(\alpha_p-\alpha_e)$$

From this, we see that the free fall time for a heavy particle (proton) is longer than that for a light particle (electron). For the proton and electron the relation between their fall times is

$$\frac{t_p-T}{t_e-T}=\frac{\alpha_p}{\alpha_e}=\left(\frac{m_p}{m_e}\right)^2=3.4\times 10^6 \tag{93}$$

Indeed, if quantum space-time structure plays a role as a fifth force in processes in the microworld, then the free-fall time relation of microparticles may be measured experimentally using relations of the type of (93). However, the absolute value of t_q-T is very small and depends on the value of the fundamental length. For example, let a quasiclassical object with mass of the order of the Planck mass $m_{\text{Pl}} \sim 10^{-5}$ g fall from height $H = 100$ cm; then its falling time becomes longer with respect to the classical one $T = (2H/g)^{1/2}$ according to the following formulas:

$$\Delta t = 1.33 \times 10^{-18}\ \text{sec} \qquad \text{if } L \sim 10^{-33}\ \text{cm}$$

$$\Delta t = 1.33 \times 10^{-12}\ \text{sec} \qquad \text{if } L \sim 10^{-30}\ \text{cm}$$

$$\Delta t = 1.33 \times 10^{-6}\ \text{sec} \qquad \text{if } L \sim 10^{-27}\ \text{cm}$$

4.2. Change of Time Scale in Quantum Space–Time

Consider clocks moving with an arbitrary velocity in a quantum gravitational field. Then, according to the above result, in a quantum coordinate system, the space-time interval between counts shown by the clocks is given by formula (57),

$$\Delta\tau = \left[g_{nm}(x)\, dx^n\, dx^m + 4iL^2\sigma_{ab} g_{nm}(x) \frac{\partial e_a^\rho}{\partial x^\delta} \frac{\partial e_b^n}{\partial x^\rho}\, dx^\delta\, dx^m \right]^{1/2}$$

Since the velocity of the clocks is dx^μ/dt, the time interval between counts is defined by

$$\frac{dt}{\Delta\tau} = \left[g_{nm}(x) \frac{dx^n}{dt} \frac{dx^m}{dt} + 4iL^2\sigma_{ab} g_{nm}(x) \frac{\partial e_a^\rho}{\partial x^\delta} \frac{\partial e_b^n}{\partial x^\rho} \frac{dx^\delta}{dt} \frac{dx^m}{dt} \right)^{-1/2} \tag{94}$$

In particular, if the clocks are at rest, one gets

$$\frac{dt}{\Delta\tau} = \left[g_{00}(x) + 4i\sigma_{ab} L^2 g_{n0}(x) \frac{\partial e_a^\rho}{\partial x^0} \frac{\partial e_b^n}{\partial x^\rho} \right]^{-1/2} \tag{95}$$

As in the usual theory of gravity, in the quantum theory we do not observe the coefficients of change of the time scale appearing in (94) and (95) by measuring the time interval dt between two counts and comparing it with the averaged value $\langle\Delta\tau\rangle$, where $\langle\cdot\cdot\cdot\rangle$ means some averaging procedure determined below. However, we can compare the coefficients of change of the time scale due to the quantum nature of space-time at two different points of the field. It is assumed, for example, that at point 1 we observe a light signal coming from point 2, where it appears as a result of some atomic transition. Therefore, according to formula (95), the time between two successive signals arriving at point 1 will be connected with the time between those leaving from point 2 by the formula

$$dt_2 = \langle\Delta\tau\rangle\langle[g_{00}(x_2) + f(x_2)]^{-1/2}\rangle$$

where

$$f(x) = 4iL^2\sigma_{ab} g_{n0}(x) \frac{\partial e_a^\rho}{\partial x^0} \frac{\partial e_b^n}{\partial x^\rho}$$

If an analogous atomic transition takes place at the point 1, then the time separating the arriving light wave signals measured at point 1 is equal to

$$dt_1 = \langle \Delta \tau \rangle \langle [g_{00}(x_1) + f(x_1)]^{-1/2} \rangle$$

Thus, for the given atomic transition, the ratio of frequencies for (observing at point 1) light leaving from point 2 and light coming from point 1 is given by

$$\frac{\nu_2}{\nu_1} = \left\langle \left[\frac{g_{00}(x_2) + f(x_2)}{g_{00}(x_1) + f(x_1)} \right]^{1/2} \right\rangle$$
$$= \left[\frac{g_{00}(x_2)}{g_{00}(x_1)} \right]^{1/2} \left\langle \left\{ 1 + \frac{1}{2} \left[\frac{f(x_2)}{g_{00}(x_2)} - \frac{f(x_1)}{g_{00}(x_1)} \right] \right\} \right\rangle$$

For the limiting case of weak field, $g_{00} \approx -1 - 2\phi$ and $|\phi| \ll 1$, so that $\nu_2/\nu_1 = 1 + \Delta\nu/\nu$, where

$$\Delta \nu / \nu = [\phi(x_2) - \phi(x_1)] \{ 1 + \tfrac{1}{2} \langle [f(x_1) - f(x_2)] \rangle \} \tag{96}$$

Now it remains to calculate the additional term $f(x)$ in (95) caused by quantum space-time. In two-dimensional space-time for the simple case (5) we have

$$\sigma_{ab} g_{n0}(x) \frac{\partial e_a^\rho}{\partial x^0} \frac{\partial e_b^n}{\partial x^\rho} = \sigma_{ab} g_{00} \frac{\partial e_a^\rho}{\partial x^0} \frac{\partial e_b^0}{\partial x^\rho} + \sigma_{ab} g_{10} \frac{\partial e_a^\rho}{\partial x^0} \frac{\partial e_b^1}{\partial x^\rho} \tag{97}$$

The first term is equal to

$$I_1 = \sigma_{ab} g_{00} \frac{\partial e_a^\rho}{\partial x^0} \frac{\partial e_b^0}{\partial x^\rho}$$
$$= \sigma_{10} g_{00} \frac{\partial e_1^0}{\partial x^0} \frac{\partial e_0^0}{\partial x^0} + \sigma_{10} g_{00} \frac{\partial e_1^1}{\partial x^0} \frac{\partial e_0^0}{\partial x}$$
$$+ \sigma_{01} g_{00} \frac{\partial e_0^0}{\partial x^0} \frac{\partial e_1^0}{\partial x^0} + \sigma_{01} g_{00} \frac{\partial e_0^1}{\partial x^0} \frac{\partial e_1^0}{\partial x}$$

Then, taking into account (5), we get

$$I_1 = g_{00} \left[\sigma_{10} \left(-\frac{\omega}{c} \right) \left(\frac{\omega}{c} \right) \sin \delta \cos \delta + \sigma_{10} \left(-\frac{\omega k}{c} \right) \cos^2 \delta \right.$$
$$\left. + \sigma_{01} \left(-\frac{\omega^2}{c^2} \right) \sin \delta \cos \delta + \sigma_{01} \left(\frac{\omega k}{c} \right) \sin^2 \delta \right]$$

where

$$\sigma_{10} = -\sigma_{01} = -\gamma_0 \gamma_1 / i, \qquad \delta = \omega t - kx$$

Thus,

$$I_1 = (1/i)\gamma_0\gamma_1(\omega k/c)g_{00}$$

The analogous calculation for (97) gives

$$\begin{aligned} I_2 &= \sigma_{ab}g_{10}\frac{\partial e_a^\rho}{\partial x^0}\frac{\partial e_b^1}{\partial x^\rho} \\ &= \sigma_{10}g_{10}\frac{\partial e_1^0}{\partial x^0}\frac{\partial e_0^1}{\partial x^0}+\sigma_{10}g_{10}\frac{\partial e_1^1}{\partial x^0}\frac{\partial e_0^1}{\partial x} \\ &\quad +\sigma_{01}g_{10}\left(\frac{\partial e_0^0}{\partial x^0}\frac{\partial e_1^1}{\partial x^0}+\frac{\partial e_0^1}{\partial x^0}\frac{\partial e_1^1}{\partial x}\right) \end{aligned}$$

or in explicit form

$$I_2 = \sigma_{01}g_{10}(x)(\omega^2/c^2)$$

Adding the obtained results, one obtains for light signals $(k=\omega/c)$

$$I_1+I_2 = \frac{\omega^2}{c^2}\frac{\gamma_0\gamma_1}{i}g_{00}^{(x)}\left[1+\frac{g_{10}(x)}{g_{00}(x)}\right]$$

Thus, in the case of the simple formula (5), the expression (95) takes the form

$$\frac{dt}{\Delta\tau} = [g_{00}(x)]^{-1/2}\left\{1+4\frac{\omega^2}{c^2}\gamma_0\gamma_1 L^2\left[1+\frac{g_{10}(x)}{g_{00}(x)}\right]\right\}^{-1/2}$$

or

$$\frac{dt}{\Delta\tau} = [g_{00}(x)]^{-1/2}\left\{1-2\frac{\omega^2}{c^2}L^2\gamma_0\gamma_1\left[1+\frac{g_{10}(x)}{g_{00}(x)}\right]\right\}$$

Here, one should use some averaging procedure, which reduces to taking the expectation value over the vacuum-like state (66) in the quantum space-time case. Thus,

$$\frac{dt}{\langle\Delta\tau\rangle} = [g_{00}(x)]^{-1/2}\left\{1-\frac{2\omega^2}{c^2}L^2\left[1+\frac{g_{10}(x)}{g_{00}(x)}\right]\right\} \tag{98a}$$

In this particular case, the relation (96) has the form

$$\frac{\Delta\nu}{\nu} = [\phi(x_2)-\phi(x_1)]\left\{1+\frac{2\omega^2L^2}{c^2}\left[\frac{g_{10}(x_2)}{g_{00}(x_2)}-\frac{g_{10}(x_1)}{g_{00}(x_1)}\right]\right\} \tag{98b}$$

From this we see that the contribution to the value of the gravitational red shift due to quantum space-time structure is very small (even zero, when one takes the trace of γ^a matrices) of the order $L^2\phi(x)$, since we assume that $|g_{10}(x)| = |h_{10}(x)| \sim |\phi(x)| \ll 1$, where $\phi(x)$ is the gravitational potential of the body from which light arrives at the observer.

Up to now we have not discussed the value of ω in (98a), which depends on the concrete properties of the system (atomic transition) that emitted the light. Generally speaking, ω is connected with the emitting and receiving frequencies of ν_1 and ν_2 at points 1 and 2, respectively. In the last case, the ratio (98b) acquires the form

$$\frac{\Delta\nu}{\nu}=[\phi(x_2)-\phi(x_1)]\left[1+\frac{16\pi^2L^2}{c^2}\nu_1^2\left(\frac{\Delta\nu}{\nu}\right)\right]$$

However, there is one interesting experiment testing the gravitational red shift, realized by Pound and Rebka (1960). They allowed a photon emitted by ^{57}Fe due to an energy transition of 14.4 keV (0.1 mks) to fall from a height of 22.6 m and observed its resonance absorption by the same atom ^{57}Fe. For this experimental situation, for the quantity ω we choose $\omega \to \omega_0 = \varepsilon/\hbar$, with $\varepsilon = 14.4$ keV. For this

$$\nu_2=[g_{00}(x_2)]^{1/2}\Big/\left(1-2\frac{\omega_0^2}{c^2}L^2\right)$$

for the light source ^{57}Fe situated at point 2, and $\nu_1=[g_{00}(x_1)]^{1/2}$ for the observation point (target) 1. Then, in this concrete case, we have the ratio

$$\frac{\Delta\nu}{\nu}=[\phi(x_2)-\phi(x_1)]\left(1+2\frac{\omega_0^2}{c^2}L^2\right) \tag{99}$$

In the usual theory of gravity, if the equivalence principle is valid, one must expect that the light frequency falling into the target will be shifted by the classical value

$$(\Delta\nu/\nu)_{\rm cl}=-\Delta\phi=\phi(x_1)|_{\rm target}-\phi(x_2)|_{\rm source}=2.46\times10^{-15}$$

At present, this theoretical calculation coincides with experimental data $(\Delta\nu/\nu)_{\rm exp}=2.6\times10^{-15}$ to an accuracy of about 1% (Pound and Snider, 1964). Therefore, the quantum correction in (99) should be less than experimental errors:

$$2.6\times10^{-15}\frac{2L^2\omega_0^2}{c^2}\lesssim0.26\times10^{-16}, \qquad \omega_0=\frac{\varepsilon}{h}$$

From this we conclude that

$$L\lesssim2\times10^{-10}\ \text{cm}$$

Thus, we have shown that the change of time scale due to the quantum nature of space-time is quite small. However, there is a more sensitive effect of the structure of space-time at small distances, which we now discuss.

4.3. Relativity and the Anisotropy of Inertia

According to the above considerations, the quantum nature of space-time at small distances, after averaging over the large scale, plays a role in the formation of some anisotropy of the universe and, in turn, gives rise to a slight change of the laws of motion in inertial systems of reference. It is natural to assume that the appearance of anisotropy is caused by the additional force obtained in Section 2. In other words, this force may be understood as the source of a small difference in the values of gravitational and inertial masses.

On the level of the usual theory of gravity, in connection with the verification of Mach's principle of the possible influence of large mass accumulations (for example, in the presence of the Milky Way) on the laws of motion, experiments (Hughes et al., 1960; Drever, 1961) devoted to testing the existence of a small difference in inertial mass have been carried out. Hughes and his team observed resonance absorption of photons by ^{7}Li nuclei in a magnetic field. The experimental result is that if one can present a nucleus of ^{7}Li as a single proton with angular momentum $J = \frac{3}{2}$, which is connected with other nucleons in a central symmetric potential, then the anisotropy of the proton mass Δm must be equal to

$$\Delta\left(\frac{p^2}{2m}\right) \approx \frac{\Delta m}{m}\left(\frac{p^2}{2m}\right) \lesssim 5.3 \times 10^{-21}\ \text{MeV} \tag{100}$$

where $p^2/2m$ is the kinetic energy of the proton. Since $p^2/2m$ is larger than $\frac{1}{2}$ MeV, this is reduced to the assertion that the anisotropy of inertial mass is bounded by

$$(\Delta m/m) \lesssim 10^{-20} \tag{101}$$

We know that in quantum space-time the kinetic energy of the particle is changed in accordance with formulas (12) and (18). This in turn gives an additional energy shift (30) for the atomic level in the stationary case. We assume that this change of energy level in ^{7}Li is connected with anisotropy of the proton mass given by (100) or (101).

Thus, first we write the change of kinetic energy due to the quantum nature of space-time by means of the anisotropy of inertial mass,

$$\frac{p^2}{2m} \Rightarrow \frac{P^2}{2m} = \frac{p^2}{2(m-\Delta m)} = \frac{p^2}{2m}\left(1 + \frac{\Delta m}{m}\right)$$

Second, this change is connected with the shift of atomic energy level given by (30). For the given case of ^{7}Li ($J = +\frac{1}{2}$, $l = 1$, $n = 2$, $Z = 6$), we get

$$\frac{\Delta m}{m} = Z^4\left(\frac{L}{a}\right)^2 \delta^3 \frac{hRy}{6(p^2/2m)} \tag{102}$$

where $hcRy = m_e e^4 hc/4\pi\hbar^3 = 13.6$ eV, $a = 0.529 \times 10^{-8}$ cm is the Bohr radius, and $\delta = m_p/m_e$. On the other hand, the relation (102) is bounded by the experimental value (101). Therefore, one can obtain the following estimation for the value of the fundamental length:

$$L \lesssim 9.02 \times 10^{-23} \text{ to } \sim 10^{-22} \text{ cm}$$

Thus, we see that the anisotropy property of inertia is very sensitive to the quantum structure of space-time at small distances. Of course, the latter gives rise to the appearance of the slight anisotropy of the universe.

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